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# Allocation of shared costs among decision making units: a DEA approach

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## Abstract

An issue of considerable importance, both from a practical organizational standpoint and from a costs research perspectives, involves the allocation of fixed resources or costs across a set of competing entities in an equitable manner. Cook and Kress (Eur. J. Oper. Res. 119 (1999) 652) propose a data envelopment analysis (DEA) approach to obtain a theoretical framework for such cost allocation problems. Their approach cannot be used directly to determine a cost allocation among the decision making units (DMUs), but rather to examine existing costing rules for equity. The current paper extends the Cook and Kress (Eur. J. Oper. Res. 119 (1999) 652) approach, and provides a practical approach to the cost allocation problem. It is shown that an equitable cost allocation can be achieved using DEA principles.

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## 1. Introduction

An issue of considerable importance, both from a practical organizational standpoint and from a costs research perspective, involves the allocation of fixed resources or costs of a set of competing entities in an equitable manner. An example is the allocation of a manufacturer's advertising expenditures onto local retailers. Cook and Kress [1] propose a data envelopment analysis (DEA) approach to obtain this kind of cost allocation. Their theoretical foundation is based upon two assumptions: invariance and pareto-minimality. While their method is a natural extension of the simple one-dimensional problem to the general multiple-input multiple-output case, no executable approach is provided to determine a set of such cost allocation. As indicated in Cook and Kress [1], their

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approach is not intended to be used directly to determine a cost allocation among the decision making units (DMUs), but rather to examine existing costing rules for equity. Their approach is also based upon an output-oriented DEA model whose frontier exhibits constant returns to scale (CRS).

Beasley [2] provide an alternative DEA-based cost allocation approach by maximizing the average efficiency across all DMUs and adding additional constraints and models to obtain a unique cost allocation. We note that Cook and Kress [1] and Beasley [2] are two very different approaches, because the underlying assumptions are different. The former assumes that the current DEA efficiency remain unchanged after the cost allocation while the latter assumes that the average DEA efficiency of all DMUs is maximized after the cost allocation, i.e., the original DEA efficiency can be changed.

Apparently, there are many feasible cost allocations to Cook and Kress [1]. We are only interested in finding one cost allocation. The current paper extends the results in Cook and Kress [1] into other DEA models with different orientations. While Cook and Kress [1] provide a theoretical framework for examining cost allocation problems, the current paper builds upon this idea to provide a practical approach wherein cost allocations can actually be achieved under DEA.

The following section provides basic DEA models and summarizes the results in Cook and Kress [1]. Section 3 extends their model to enable the allocation of costs. This new approach is illustrated in Section 4 with the numerical example in Cook and Kress [1]. The final section presents concluding remarks.

## 2. Background

Suppose we have a set of units,  $DMU_j$ , ( $j = 1, \dots, n$ ). Each DMU uses  $m$  inputs  $x_{ij}$  ( $i = 1, \dots, m$ ) to produce  $s$  outputs  $y_{rj}$  ( $r = 1, \dots, s$ ). Then the (relative) efficiency of  $DMU_j$  can be expressed as

$$E_j = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}},$$

where  $u_r$  and  $v_i$  are (unknown) output and input multipliers, respectively. In DEA,  $E_j$  is obtained by solving the following CCR ratio model [3], when information on  $u_r$  and  $v_i$  is not available.

$$\begin{aligned} \max \quad & \frac{\sum_{r=1}^s u_r y_{rj_o}}{\sum_{i=1}^m v_i x_{ij_o}} \\ \text{s.t.} \quad & \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1, \quad \forall j \\ & u_r, v_i \geq 0 \end{aligned} \tag{1}$$

where  $j_o$  represents one of the DMUs,  $DMU_{j_o}$ . Model (1) is usually referred to as the input-oriented CRS DEA model.

If the relative efficiency is defined as  $\frac{\sum_{i=1}^m v_i x_{ij}}{\sum_{r=1}^s u_r y_{rj}}$ , then the associated output-oriented DEA model is

$$\begin{aligned} \min \quad & \frac{\sum_{i=1}^m v_i x_{ij_o}}{\sum_{r=1}^s u_r y_{rj_o}} \\ \text{s.t.} \quad & \frac{\sum_{i=1}^m v_i x_{ij}}{\sum_{r=1}^s u_r y_{rj}} \geq 1, \quad \forall_j \\ & u_r, v_i \geq 0. \end{aligned} \tag{2}$$

Suppose that a cost  $R$  is to be distributed among the  $n$  DMUs. That is, each DMU is to be allocated a cost  $r_j$  such that

$$\sum_{j=1}^n r_j = R.$$

If this  $r_j$  is treated as a new input, then the efficiency becomes

$$E_j^R = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij} + v r_j} \left( \text{or } \frac{\sum_{i=1}^m v_i x_{ij} + v r_j}{\sum_{r=1}^s u_r y_{rj}} \right).$$

In the Cook and Kress model, it is assumed that  $R$  will be assigned in such a way that the relative efficiencies of DMUs remain unchanged. Specifically, they adopt an *invariance* assumption,  $E_j = E_j^R$ . The authors observe that due to the optimization procedures, however, it is permissible for  $v = 0$ . As a result,  $R$  can be distributed in its entirety among only the inefficient DMUs in any proportion whatever, meaning that the DEA efficiency ratings would not change, and the invariance assumption would be satisfied. However, any allocation which penalizes only the inefficient DMUs, would generally be unacceptable to the organization. Thus, Cook and Kress [1] impose the *pareto-minimality* condition which does not permit the cost allocation only among inefficient DMUs.

Using these two assumptions, Cook and Kress [1] develop a theoretical framework for the cost allocation, based upon model (2). In the multiple inputs and multiple outputs case, their approach obtains a characterization for an equitable allocation of shared costs. However, as pointed out by Cook and Kress [1], this characterization cannot be used to directly determine a cost allocation among the DMUs, but rather it serves as a means of examining existing costing rules for equity.

In the following section, we use DEA principles to develop a procedure that can be used to derive a cost allocation among the  $n$  DMUs. Our procedure also enables us to consider the cost allocation issue under other DEA models with different orientations, e.g., model (1).

### 3. A practical DEA approach to fixed cost allocation

#### 3.1. Output-oriented CRS cost allocation

An output-oriented DEA model where inputs are fixed at their current levels while maximizing the output levels, i.e., model (2), is used in Cook and Kress [1]. Here, we consider the following

DEA model which is the linear programming model equivalent to model (2):

$$\begin{aligned}
 E_{j_o} &= \max \phi_{j_o} \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ij_o}, \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \phi_{j_o} y_{rj_o}, \quad r = 1, 2, \dots, s, \\
 & \lambda_j \geq 0.
 \end{aligned} \tag{3}$$

Intuitively, to obtain  $E_{j_o}^R$ , we would apply model (3) with the additional input of  $r_j$ . However, because of the pareto-minimality condition which does not permit the cost allocation only among inefficient DMUs,  $E_{j_o}^R$  should be calculated as the optimal solution to the following linear programming model:

$$\begin{aligned}
 E_{j_o}^R &= \max \tilde{\phi}_{j_o} \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ij_o}, \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j r_j = r_{j_o}, \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \tilde{\phi}_{j_o} y_{rj_o}, \quad r = 1, 2, \dots, s, \\
 & \lambda_j \geq 0.
 \end{aligned} \tag{4}$$

Note that  $\sum_{j=1}^n \lambda_j r_j \leq r_{j_o}$  is replaced by  $\sum_{j=1}^n \lambda_j r_j = r_{j_o}$ . Referring to Cook and Kress [1], this equation arises from the requirement that the reduced cost is to be non-negative for the new input variable,  $r_{j_o}$ . In fact, the expression  $-r_{j_o} + \sum_{j=1}^n \lambda_j r_j$  is the reduced cost for that variable. This equation excludes the possible inefficiency (non-zero DEA slack) from the cost allocation. Note also that for a non-frontier (inefficient) DMU,  $\sum_{j \in F} \lambda_j^{j_o*} r_j = r_{j_o}$ , where  $F$  represents the set of frontier (efficient) DMUs. Because some  $\lambda_j^{j_o*}$  must be positive,  $\sum_{j=1}^n \lambda_j r_j = r_{j_o}$  ensures that cost allocation will not be entirely distributed among inefficient DMUs.

If  $DMU_{j_o}$  is a frontier DMU, then  $E_{j_o} = \phi_{j_o}^* = \tilde{\phi}_{j_o}^* = 1$ . Suppose  $DMU_{j_o}$  is not a frontier DMU, then we have  $\phi_{j_o}^* > 1$  with a set of optimal  $\lambda_j^{j_o*}$  for model (3). Now if  $\sum \lambda_j^{j_o*} r_j = r_{j_o}$ , then  $\lambda_j^{j_o*}$  and  $\phi_{j_o}^*$  are also optimal in model (4). As a result,  $E_{j_o} = \phi_{j_o}^* = \tilde{\phi}_{j_o}^* = E_{j_o}^R > 1$ .

Let  $N$  represents the set of non-frontier DMUs. Assume we have a cost allocation of  $r_j$  ( $j = 1, \dots, n$ ), then  $\sum_{j \in F} \lambda_j^t r_j = r_t$  for all  $t \in N$ . This relationship based upon the optimal solutions in model (3), satisfies the invariance assumption and does not allow the cost allocation only among inefficient DMUs.

Consequently, this cost allocation can be obtained by solving the following linear programming problem with an arbitrary objective function  $P$ :

$$\begin{aligned}
 & \min \quad P \\
 & \text{s.t.} \quad \sum_{j \in F} \lambda_j^{t*} r_j = r_t \quad t \in N, \\
 & \quad \quad \sum_{j=1}^n r_j = R,
 \end{aligned} \tag{5}$$

where  $\lambda_j^{t*}$  are optimal values in model (3) when non-frontier DMUs ( $t \in N$ ) are under evaluation.

### 3.2. Input-oriented CRS cost allocation

Consider now the cost allocation problem using the input-oriented CRS DEA model, i.e., model (1). In this case,  $E_{j_o}$  is calculated using the following model which is equivalent to model (1)

$$\begin{aligned}
 E_{j_o} &= \min \quad \theta_{j_o} \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_{j_o} x_{ij_o}, \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rj_o}, \quad r = 1, 2, \dots, s, \\
 & \lambda_j \geq 0.
 \end{aligned} \tag{6}$$

$E_{j_o}^R$  is calculated using the following linear programming problem:

$$\begin{aligned}
 E_{j_o}^R &= \min \quad \tilde{\theta}_{j_o} \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \tilde{\theta}_{j_o} x_{ij_o}, \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j r_j = r_{j_o}, \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rj_o}, \quad r = 1, 2, \dots, s, \\
 & \lambda_j \geq 0.
 \end{aligned} \tag{7}$$

where  $r_j$  satisfies  $\sum_{j \in F} \lambda_j^{t*} r_j = r_t$  for all  $t \in N$  and  $\lambda_j^{t*}$  are optimal values in model (6).

Again the  $\sum_{j=1}^n \lambda_j r_j = r_{j_o}$  in model (7) ensures that the cost allocation does not occur only in inefficient DMUs. Now, if  $DMU_{j_o}$  is a frontier DMU, then  $E_{j_o} = E_{j_o}^R = 1$ . Next, suppose  $DMU_{j_o}$  is not a frontier DMU, then  $\theta_{j_o}^* < 1$  with a set of optimal  $\lambda_j^{j_o*}$  in model (6). If the optimal  $\lambda_j^{j_o*}$  in model (6) satisfy  $\sum \lambda_j^{j_o*} r_j = r_{j_o}$ , then  $\lambda_j^{j_o*}$  and  $\theta_{j_o}^*$  are also optimal in model (7). Thus,  $E_{j_o} = \theta_{j_o}^* = \tilde{\theta}_{j_o}^* = E_{j_o}^R < 1$ .

Therefore, we can use model (5) with optimal values of  $\lambda_j^{j*}$  obtained from model (6) to get a cost allocation. Note that model (7) is actually a DEA model where the fixed cost is treated as an uncontrollable or non-discretionary input. This, in fact, reflects the real situation of the cost allocation, since the DMUs themselves do not have control over the fixed cost.

#### 4. Illustration

Table 1 presents the numerical example used in Cook and Kress [1] where we have 12 DMUs, 3 inputs and 2 outputs. As in Beasley [2], we suppose that we have a fixed cost of 100 to be allocated. Five DMUs are frontier DMUs with a score of one and seven are non-frontier with a score greater than one based upon model (3) (see the 7th column in Table 1).

Table 2 reports the optimal  $\lambda$  for non-frontier DMUs from model (3). For example, when DMU1 is under evaluation by model (3), the frontier DMUs are DMU8 (with  $\lambda_8^* = 0.52$ ), and DMU9 (with  $\lambda_9^* = 0.64$ ).

Table 1  
Sample DMUs

DMU	Input1	Input2	Input3	Output1	Output2	Efficiency	Fixed cost
1	350	39	9	67	751	1.32	11.22
2	298	26	8	73	611	1.08	0
3	422	31	7	75	584	1.34	16.95
4	281	16	9	70	665	1	0
5	301	16	6	75	445	1	0
6	360	29	17	83	1070	1.04	15.43
7	540	18	10	72	457	1.16	0
8	276	33	5	78	590	1	0
9	323	25	5	75	1074	1	17.62
10	444	64	6	74	1072	1.20	21.15
11	323	25	5	25	350	3	17.62
12	444	64	6	104	1199	1	0

Table 2  
Optimal  $\lambda$

		DMU		DMU		DMU
DMU1	0.52	8	0.64	9		
DMU2	0.50	4	0.04	5	0.53	8
DMU3	0.32	5	0.06	8	0.96	9
DMU6	0.12	4	0.16	8	0.88	9
DMU7	0.14	4	0.99	5		
DMU10	1.2	9				
DMU11	1	9				

Based upon Table 2 and model (5), we obtain a cost allocation (see the last column in Table 1). It can be easily seen that this cost allocation satisfies the invariance and pareto-minimality conditions in Cook and Kress [1].

Beasley [2] also used this numerical example to obtain a unique cost allocation,  $r_1 = 6.78$ ,  $r_2 = 7.21$ ,  $r_3 = 6.83$ ,  $r_4 = 8.47$ ,  $r_5 = 7.08$ ,  $r_6 = 10.06$ ,  $r_7 = 5.09$ ,  $r_8 = 7.74$ ,  $r_9 = 15.11$ ,  $r_{10} = 10.08$ ,  $r_{11} = 1.58$  and  $r_{12} = 13.97$ . It is easy to verify that all DMUs become efficient when this cost allocation is used as an additional input, i.e., this cost allocation is not a feasible one under the assumptions of Cook and Kress [1] and the current paper. Thus, this numerical example indicates that the Cook and Kress [1] and Beasley [2] approaches are different.

Note that there are  $N + 1$  constraints with  $n$  variables in the model (5). As a result, model (5) does not yield a unique solution.<sup>1</sup> However, we are only interested in obtaining a feasible cost allocation. If one is interested in obtaining a unique cost allocation, one can incorporate additional constraints on  $r_j$ , e.g., cone ratio [8] type of constraints into model (5) or impose lower and upper bounds of the cost allocation as in Beasley [2]. Such a priori information will also eliminate the zero cost allocation among some DMUs if the assumption is that each DMU should have a share of cost allocation.

## 5. Conclusions

The current paper develops a DEA-based approach to cost allocation problems. As a result of the current study, we can extend the Cook and Kress [1] approach to other DEA models with different model orientations. For example, if we incorporate  $\sum_{j=1}^n \lambda_j = 1$  into models (3) and (4), we obtain a cost allocation under the condition of variable returns to scale (VRS). We leave the development to the interested reader.

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<sup>1</sup> Note that multiple optimal  $\lambda$  values are likely to occur in model (3). As shown in Zhu and Shen [4] and Banker et al. [5], this is due to the linear dependency among efficient DMUs. See also the face regularity condition in Thrall [6] and Seiford and Zhu [7] where an approach is proposed to detect the situation.

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