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Interfaces with Other Disciplines

Rank order data in DEA: A general framework

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Abstract

In data envelopment analysis (DEA), performance evaluation is generally assumed to be based upon a set of quantitative data. In many real world settings, however, it is essential to take into account the presence of qualitative factors when evaluating the performance of decision making units (DMUs). Very often rankings are provided from best to worst relative to particular attributes. Such rank positions might better be presented in an ordinal, rather than numerical sense. The paper develops a general frame work for modeling and treating qualitative data in DEA and provides a unified structure for embedding rank order data into the DEA framework. The existing techniques are discussed and their equivalence is demonstrated. Both continuous and discrete projection models are provided. It is shown that qualitative data can be treated in conventional DEA methodology.

Keywords: Data envelopment analysis (DEA); Efficiency; Qualitative; Rank order data

1. Introduction

In the data envelopment analysis (DEA) model of Charnes et al. (1978), each member of a set of decision making units (DMUs) is to be evaluated relative to its peers. This evaluation is generally assumed to be based on a set of *quantitative* output and input factors. In many real world settings, however, it is essential to take into account the presence of *qualitative* factors when rendering a decision on the performance of a DMU. Very often it is the case that for a factor such as management competence, one can, at most, provide a *ranking* of the DMUs from best to worst relative to this attribute. The capability of providing a more

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precise, quantitative measure reflecting such a factor is generally beyond the realm of reality. In some situations such factors can be legitimately 'quantified', but very often such quantification may be superficially forced as a modeling convenience.

In situations such as that described, the 'data' for certain influence factors (inputs and outputs) might better be represented as rank positions in an ordinal, rather than numerical sense. Refer again to the management competence example. In certain circumstances, the information available may permit one *only* to put each DMU into one of L categories or groups (e.g. 'high', 'medium' and 'low' competence). In other cases, one may be able to provide a complete rank ordering of the DMUs on such a factor.

Cook et al. (1993, 1996) first presented a modified DEA structure, incorporating rank order data. The 1996 article applied this structure to the problem of prioritizing a set of research and development projects, where both inputs and outputs were defined on a Likert scale. An alternative to the Cook et al approach was provided in Cooper et al. (1999) in the form of the imprecise DEA (IDEA) model. While various forms of imprecise data were examined, one major component of that research focused on ordinal (rank order) data.

In the current paper, we present a unified structure for embedding rank order or Likert scale data into the DEA framework. To provide a practical setting for the methodology to be developed herein, Section 2 briefly discusses the R&D project selection problem as presented in more detail in Cook et al. (1996), and the Korean Telephone offices problem of Kim et al. (1999). Section 3 presents a continuous projection model, based on the conventional radial model of Charnes et al. (1978). In Section 4 this approach is compared to the IDEA methodology of Cooper et al. (1999). We demonstrate that IDEA for Likert scale data is in fact equivalent to the earlier approach of Cook et al. (1996). Section 5 develops a discrete projection methodology that guarantees projection to points on the Likert scale. Conclusions and further directions are addressed in Section 6.

2. Applications

2.1. Ordinal data in R&D project selection

Consider the problem of selecting R&D projects in a major public utility corporation with a large research and development branch. Research activities are housed within several different divisions, for example, thermal, nuclear, electrical, and so on. In a budget constrained environment in which such an organization finds itself, it becomes necessary to make choices among a set of potential research initiatives or projects that are in competition for the limited resources. To evaluate the impact of funding (or not funding) any given research initiative, two major considerations generally must be made. First, the initiative must be viewed in terms of more than one factor or criterion. Second, some or all of the criteria that enter the evaluation may be qualitative in nature. Even when clearly quantitative factors are involved, such as long term saving to the organization it may be extremely difficult to obtain even a crude estimate of the value of that factor. The most that one can do in many such situations is to classify the project (according to this factor) on some scale (high/medium/low or say a 5-point scale).

Let us assume that for each qualitative criterion each initiative is rated on a 5-point scale, where the particular point on the scale is chosen through a consensus on the part of executives within the organization. Table 1 presents an illustration of how the data might appear for 10 projects, 3 qualitative output criteria (benefits), and 3 qualitative input criteria (cost or resources). In the actual setting examined, a number of potential benefit and cost criteria were considered as discussed in Cook et al. (1996).

We use the convention that for both outputs and inputs, a rating of 1 is 'best', and 5 'worst'. For outputs, this means that a DMU ranked at position 1 generates *more* output than is true of a DMU in position 2, and so on. For inputs, a DMU in position 1 consumes *less* input than one in position 2.

Table 1
Ratings by criteria

Project no.	Outputs			Inputs		
	1	2	3	4	5	6
1	2	4	1	5	2	1
2	1	1	4	3	5	2
3	1	1	1	1	2	1
4	3	3	3	4	3	2
5	4	3	5	5	1	4
6	2	5	1	1	2	2
7	1	4	1	5	4	3
8	1	5	3	3	3	3
9	5	2	4	4	2	5
10	5	4	4	5	5	5

Regardless of the manner in which such a scale rating is arrived at, the existing DEA model is capable only of treating the information as if it has cardinal meaning (e.g. something which receives a score of 4 is evaluated as being twice as important as something that scores 2). There are a number of problems with this approach. First and foremost, the projects' original data in the case of some criteria may take the form of an ordinal ranking of the projects. Specifically, the most that can be said about two projects *i* and *j* is that *i* is preferred to *j*. In other cases it may only be possible to classify projects as say 'high', 'medium' or 'low' in importance on certain criteria. When projects are rated on, say, a 5-point scale, it is generally understood that this scale merely provides a relative positioning of the projects. In a number of agencies investigated (for example, hydro electric and telecommunications companies), 5-point scales are common for evaluating alternatives in terms of qualitative data, and are often accompanied by interpretations such as

1 = Extremely important
2 = Very important
3 = Important
4 = Low in importance
5 = Not important

which are easily understood by management. While it is true that market researchers often treat such scales in a numerical (i.e. cardinal) sense, no one seriously believes that an 'extremely important' classification for a project should be interpreted literally as meaning that this project rates three times better than one which is only classified as 'important.' The key message here is that many, if not all criteria used to evaluate R&D projects are qualitative in nature, and should be treated as such. The model presented in the following sections extends the DEA idea to an ordinal setting, hence accommodating this very practical consideration.

2.2. Efficiency performance of Korean telephone offices

Kim et al. (1999) examine 33 telephone offices in Korea and use the following factors to develop performance measures.

Inputs

(1) manpower
(2) operating costs
(3) number of telephone lines

Table 2
Data for telephone offices

DMU no.	X1	X2	X3	Y1	Y2	Y3	Y4	Y5
1	239	7.03	158	47.1	16.67	34	28	2
2	261	3.94	163	37.5	14.11	20	26	3
3	170	2.1	90	20.7	6.8	12.6	19	3
4	290	4.54	201	41.8	11.07	6.27	23	4
5	200	3.99	140	33.4	9.81	6.49	30	2
6	283	4.65	214	42.4	11.34	5.16	21	4
7	286	6.54	197	47	14.62	13	9	2
8	375	6.22	314	55.5	16.39	7.31	14	1
9	301	4.82	257	49.2	16.15	6.33	8	3
10	333	6.87	235	47.1	13.86	6.51	6	2
11	346	6.46	244	49.4	15.88	8.87	18	2
12	175	2.06	112	20.4	4.95	1.67	32	5
13	217	4.11	131	29.4	11.39	4.38	33	2
14	441	7.71	214	61.2	25.59	33	16	3
15	204	3.64	163	32.3	9.57	3.65	15	4
16	216	2.24	154	32.8	11.46	9.02	25	2
17	347	5.65	301	59	17.82	8.19	29	1
18	288	4.66	212	42.3	14.52	7.33	24	4
19	185	3.37	178	33	9.46	2.91	7	2
20	242	5.12	270	65.1	24.57	20.7	17	1
21	234	2.52	126	31.6	8.55	7.27	27	2
22	204	4.24	174	32.5	11.15	2.95	22	3
23	356	7.95	299	66	22.25	14.9	13	2
24	292	4.52	236	50	14.77	6.35	12	3
25	141	5.21	63	21.5	9.76	16.3	11	2
26	220	6.09	179	47.9	17.25	22.1	31	2
27	298	3.44	225	42.4	11.14	4.25	4	2
28	261	4.3	213	41.7	11.13	4.68	20	5
29	216	3.86	156	31.6	11.89	10.5	3	3
30	171	2.45	150	24.1	9.08	2.6	10	5
31	123	1.72	61	12	4.78	2.95	5	1
32	89	0.88	42	6.4	3.18	1.48	2	5
33	109	1.35	57	10.6	3.43	2	1	4

Outputs

(1) local revenues
(2) long distance revenues
(3) international revenues
(4) operation/maintenance level
(5) customer satisfaction

All inputs and outputs (1), (2), (3) are quantitative, and can be used in the DEA framework in the usual way. Output #4 is, however, ordinal and provides a complete ranking of the 33 DMUs. Output #5 is a categorization of the DMUs on a 5-point Likert scale. Table 2 displays the data.

3. Modeling Likert scale data: Continuous projection

The above problem typifies situations in which pure ordinal data or a mix of ordinal and numerical data are involved in the performance measurement exercise. To cast this problem in a general format, consider the situation in which a set of N decision making units (DMUs), $k = 1, \ldots, N$ are to be evaluated in terms of R_1 numerical outputs, R_2 ordinal outputs, I_1 numerical inputs, and I_2 ordinal inputs. Let $Y_k^1 = (y_{rk}^1)$, $Y_k^2 = (y_{rk}^2)$ denote the R_1-dimensional and R_2-dimensional vectors of outputs, respectively. Similarly, let $X_k^1 = (x_{ik}^1)$ and $X_k^2 = (x_{ik}^2)$ be the I_1- and I_2-dimensional vectors of inputs, respectively.

In the situation where all factors are quantitative, the conventional radial projection model for measuring DMU efficiency is expressed by the ratio of weighted outputs to weighted inputs. Adopting the general variable returns to scale (VRS) model of Banker et al. (1984), the efficiency of DMU '0' follows from the solution of

$$\begin{aligned}
e_0 = \max \quad & \left(\mu_0 + \sum_{r\in R_1} \mu_r^1 y_{r0}^1 + \sum_{r\in R_2} \mu_r^2 y_{r0}^2\right) \Bigg/ \left(\sum_{i\in I_1} \upsilon_i^1 x_{i0}^1 + \sum_{i\in I_2} \upsilon_i^2 x_{i0}^2\right) \\
\text{s.t.} \quad & \left(\mu_0 + \sum_{r\in R_1} \mu_r^1 y_{rk}^1 + \sum_{r\in R_2} \mu_r^2 y_{rk}^2\right) \Bigg/ \left(\sum_{i\in I_1} \upsilon_i^1 x_{ik}^1 + \sum_{i\in I_2} \upsilon_i^2 x_{ik}^2\right) \leqslant 1, \quad \text{all } k, \\
& \mu_r^1, \mu_r^2, \upsilon_i^1, \upsilon_i^2 \geqslant \varepsilon, \quad \text{all } r, i.
\end{aligned} \tag{3.1}$$

Problem (3.1) is convertible to the linear programming format:

$$\begin{aligned}
e_0 = \max \quad & \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{r0}^1 + \sum_{r\in R_2} \mu_r^2 y_{r0}^2 \\
\text{s.t.} \quad & \sum_{i\in I_1} \upsilon_i^1 x_{i0}^1 + \sum_{i\in I_2} \upsilon_i^2 x_{i0}^2 = 1, \\
& \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{rk}^1 + \sum_{r\in R_2} \mu_r^2 y_{rk}^2 - \sum_{i\in I_1} \upsilon_i^1 x_{ik}^1 - \sum_{i\in I_2} \upsilon_i^2 x_{ik}^2 \leqslant 0, \quad \text{all } k, \\
& \mu_r^1, \mu_r^2, \upsilon_i^1, \upsilon_i^2 \geqslant \varepsilon, \quad \text{all } r, i,
\end{aligned} \tag{3.2}$$

whose dual is given by

$$\begin{aligned}
\min \quad & \theta - \varepsilon \sum_{r\in R_1\cup R_2} s_r^+ - \varepsilon \sum_{i\in I_1\cup I_2} s_i^- \\
\text{s.t.} \quad & \sum_{k=1}^N \lambda_k y_{rk}^1 - s_r^+ = y_{r0}^1, \quad r \in R_1, \\
& \sum_{n=1}^N \lambda_k y_{rk}^2 - s_r^+ = y_{r0}^2, \quad r \in R_2, \\
& \theta x_{i0}^1 - \sum_{k=1}^N \lambda_k x_{ik}^1 - s_i^- = 0, \quad i \in I_1, \\
& \theta x_{i0}^2 - \sum_{k=1}^N \lambda_k x_{ik}^2 - s_i^- = 0, \quad i \in I_2, \\
& \sum_{k=1}^N \lambda_k = 1, \\
& \lambda_k, s_r^+, s_i^+ \geqslant 0, \quad \text{all } k, r, i, \theta \text{ unrestricted.}
\end{aligned} \tag{3.2$'$}$$

To place the problem in a general framework, assume that for each ordinal factor ($r \in R_2$, $i \in I_2$), a DMU k can be assigned to one of L rank positions, where $L \leqslant N$. As discussed earlier, $L = 5$ is an example of an appropriate number of rank positions in many practical situations. We point out that in certain application settings, different ordinal factors may have different L-values associated with them. For exposition purposes, we assume a common L-value throughout. We demonstrate later that this represents no loss of generality.

One can view the allocation of a DMU to a rank position ℓ on an output r, for example, as having assigned that DMU an output *value* or *worth* y_r^2 (ℓ). The implementation of the DEA model (3.1) (and (3.2)) thus involves determining two things:

(1) multiplier values μ_r^2, υ_i^2 for outputs $r \in R_2$ and inputs $i \in I_2$;
(2) rank position values $y_r^2(\ell)$, $r \in R_2$, and $x_{i^2}(\ell)$, $i \in I_2$, all ℓ.

In this section we show that the problem can be reduced to the standard VRS model by considering (1) and (2) simultaneously.

To facilitate development herein, define the L-dimensional unit vectors $\gamma_{rk} = (\gamma_{rk}(\ell))$, and $\delta_{ik} = (\delta_{ik}(\ell))$ where

$$\gamma_{rk}(\ell) = \begin{cases} 1 & \text{if DMU } k \text{ is ranked in } \ell\text{th position on output } r, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_{ik}(\ell) = \begin{cases} 1 & \text{if DMU } k \text{ is ranked in } \ell\text{th position on input } i, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if a 5-point scale is used, and if DMU #1 is ranked in $\ell = 3$rd place on ordinal output $r = 5$, then $\gamma_{51}(3) = 1$, $\gamma_{51}(\ell) = 0$, for all other rank positions ℓ. Thus, y_{51}^2 is assigned the value $y_5^2(3)$, the *worth* to be credited to the 3rd rank position on output factor 5. It is noted that y_{rk}^2 can be represented in the form

$$y_{rk}^2 = y_r^2(\ell_{rk}) = \sum_{\ell=1}^{L} y_r^2(\ell)\gamma_{rk}(\ell),$$

where ℓ_{rk} is the rank position occupied by DMU k on output r. Hence, model (3.2) can be rewritten in the more representative format:

$$\begin{aligned} e_0 = \max \quad & \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{r0}^1 + \sum_{r\in R_2} \sum_{\ell=1}^{L} \mu_r^2 y_r^2(\ell)\gamma_{r0}(\ell) \\ \text{s.t.} \quad & \sum_{i\in I_1} \upsilon_i^1 x_{i0}^1 + \sum_{i\in I_2} \sum_{\ell=1}^{L} \upsilon_i^2 x_i^2(\ell)\delta_{i0}(\ell) = 1, \\ & \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{rk}^1 + \sum_{r\in R_2} \sum_{\ell=1}^{L} \mu_r^2 y_r^2(\ell)\gamma_{rk}(\ell) - \sum_{i\in I_1} \upsilon_i^1 x_{ik}^1 - \sum_{i\in I_2} \sum_{\ell=1}^{L} \upsilon_i^2 x_i^2(\ell)\delta_{ik}(\ell) \leqslant 0, \text{ all } k, \\ & \{Y_r^2 = (y_r^2(\ell)), X_i^2 = (x_i^2(\ell))\} \in \Psi, \\ & \mu_r^1, \upsilon_i^1 \geqslant \varepsilon. \end{aligned} \tag{3.3}$$

In (3.3) we use the notation Ψ to denote the set of *permissible worth vectors*. We discuss this set below.

It must be noted that the same infinitesimal ε is applied here for the various input and output multipliers, which may, in fact, be measured on scales that are very different from another. If two inputs are, for example, x_{i1k}^1 representing 'labor hours', and x_{i2k}^1 representing 'available computer technology', the scales would clearly be incompatible. Hence, the likely sizes of the corresponding multipliers υ_{i1}^1, υ_{i2}^1 may be similarly different. Thrall (1996) has suggested a mechanism for correcting for such scale incompatibility, by applying a *penalty vector* G to augment ε, thereby creating differential lower bounds on the various υ_i, μ_r. Proper choice of G can effectively bring all factors to some form of common scale or unit. For simplicity of presentation we will assume the cardinal scales for all $r \in R_1$, $i \in I_1$ are similar in dimension, and that G is the unit vector. The more general case would proceed in an analogous fashion.

3.1. Permissible worth vectors

The values or worths $\{y_r^2(\ell)\}$, $\{x_i^2\ell)\}$, attached to the ordinal rank positions for outputs r and inputs i, respectively, must satisfy the minimal requirement that it is *more* important to be ranked in ℓth position than in the $(\ell+1)$st position on any such ordinal factor. Specifically, $y_r^2(\ell) > y_r^2(\ell+1)$ and $x_i^2(\ell) < x_i^2(\ell+1)$. That is, for outputs, one places a higher weight on being ranked in ℓth place than in $(\ell+1)$st place. For inputs, the opposite is true. A set of linear conditions that produce this realization is defined by the set Ψ, where

$$\begin{aligned}\Psi = \big\{(Y_r^2, X_r^2)\,|\,y_r^2(\ell) - y_r^2(\ell+1) \geqslant \varepsilon, \ell = 1,\ldots,L-1, y_r^2(L) \geqslant \varepsilon,\ x_i^2(\ell+1) - x_i^2(\ell) \geqslant \varepsilon\},\\ \ell = 1,\ldots,L-1, x_i^2(1) \geqslant \varepsilon\end{aligned}$$

Arguably, ε could be made dependent upon ℓ (i.e. replace ε by ε_ℓ). It can be shown, however, that all results discussed below would still follow. For convenience, we, therefore, assume a common value for ε. We now demonstrate that the nonlinear problem (3.3) can be written as a linear programming problem.

Theorem 3.1. *Problem (3.3), in the presence of the permissible worth space Ψ, can be expressed as a linear programming problem.*

Proof. In (3.3), make the change of variables $w_{r\ell}^1 = \mu_r^2 y_r^2(\ell)$, $w_{i\ell}^2 = \upsilon_i^2 x_i^2(\ell)$.

It is noted that in Ψ, the expressions $y_r^2(\ell) - y_r^2(\ell+1) \geqslant \varepsilon, y_r^2(L) \geqslant \varepsilon$ can be replaced by

$$\mu_r^2 y_r^2(\ell) - \mu_r^2 y_r^2(\ell+1) \geqslant \mu_r^2\varepsilon, \mu_r^2 y_r^2 \mu_r^2\varepsilon, \text{ which becomes } w_{r\ell}^1 - w_{r\ell+1}^1 \geqslant \mu_r^2\varepsilon, w_{rL}^2 \geqslant \mu_r^2\varepsilon.$$

A similar conversion holds for the $x_i^2(\ell)$. Problem (3.3) now becomes

$$\begin{aligned}
e_0 = \max \quad & \mu_0 + \sum_{r\in R_1}\mu_r^1 y_{r0}^1 + \sum_{r\in R_2}\sum_{\ell=1}^{L} w_{r\ell}^1 \gamma_{r0}(\ell)\\
\text{s.t.} \quad & \sum_{i\in I_1}\upsilon_i^1 x_{i0}^1 + \sum_{i\in I_2}\sum_{\ell=1}^{L} w_{i\ell}^2 \delta_{i0}(\ell) = 1, \quad \mu_0 + \sum_{r\in R_1}\mu_r^1 y_{rk}^1 + \sum_{r\in R_2}\sum_{\ell=1}^{L} w_{rl}^1 \gamma_{rk}(\ell),\\
& -\sum_{i\in I_1}\upsilon_i^1 x_{ik}^1 - \sum_{i\in I_2}\sum_{\ell=1}^{L} w_{i\ell}^2 \delta_{ik}(\ell) \leqslant 0, \quad \text{all } k,\\
& w_{r\ell}^1 - w_{r\ell+1}^1 \geqslant \mu_r^2\varepsilon, \quad \ell = 1,\ldots,L-1, \quad \text{all } r \in R_2,\\
& w_{rL}^1 \geqslant \mu_r^2\varepsilon, \quad \text{all } r \in R_2,\\
& w_{i\ell+1}^2 - w_{i\ell}^2 \geqslant \upsilon_i^2\varepsilon, \quad \ell = 1,\ldots,L-1, \text{ all } i \in I_2,\\
& w_{i1}^2 \geqslant \upsilon_i^2\varepsilon, \quad \text{all } i \in I_2\\
& \mu_r^1, \upsilon_i^1 \geqslant \varepsilon, \text{ all } r \in R_1, i \in I_1, \quad \mu_r^2, \upsilon_i^2 \geqslant \varepsilon, \quad \text{all } r \in R_2, i \in I_2.
\end{aligned} \tag{3.4}$$

Problem (3.4) is clearly in linear programming problem format. □

We state without proof the following theorem.

Theorem 3.2. *At the optimal solution to (3.4), $\mu_r^2 = \upsilon_i^2 = \varepsilon$ for all $r \in R_2$, $i \in I_2$.*

Problem (3.4) can then be expressed in the form:

$$
\begin{aligned}
e_0 = \max \quad & \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{r0}^1 + \sum_{r\in R_2} \sum_{\ell=1}^{L} w_{r\ell}^1 \gamma_{r0}(\ell) \\
\text{s.t.} \quad & \sum_{i\in I_1} \upsilon_i^1 x_{i0}^1 + \sum_{i\in I_2} \sum_{\ell=1}^{L} w_{i\ell}^2 \delta_{i0}(\ell) = 1, \\
& \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{rk}^1 + \sum_{r\in R_2} \sum_{\ell=1}^{L} w_{r\ell}^1 \gamma_{rk}(\ell) - \sum_{i\in I_1} \upsilon_i^1 x_{ik}^1 - \sum_{i\in I_2} \sum_{\ell=1}^{L} w_{i\ell}^2 \delta_{ik}(\ell) \leqslant 0, \quad \text{all } k, \\
& -w_{r\ell}^1 + w_{r\ell+1}^1 \leqslant -\varepsilon^2, \quad \ell = 1,\ldots,L-1, \quad \text{all } r \in R_2, \\
& -w_{rL}^1 \leqslant -\varepsilon^2, \quad \text{all } r \in R_2, \\
& -w_{i\ell+1}^2 + w_{i\ell}^2 \leqslant -\varepsilon^2, \quad \ell = 1,\ldots,L-1, \quad \text{all } i \in I_2, \\
& -w_{i1}^2 \leqslant -\varepsilon^2, \quad \text{all } i \in I_2, \\
& \mu_r^1, \upsilon_i^1 \geqslant \varepsilon, \quad r \in R_1, i \in I_1.
\end{aligned}
\tag{3.5}
$$

It can be shown that (3.5) is equivalent to the *standard* VRS model. First we form the dual of (3.5),

$$
\begin{aligned}
\min \quad & \theta - \varepsilon \sum_{r\in R_1} s_r^+ - \varepsilon \sum_{i\in I_1} s_i^- \varepsilon^2 \sum_{r\in R_2} \sum_{\ell=1}^{L} \alpha_{r\ell}^1 - \varepsilon^2 \sum_{i\in I_2} \sum_{\ell=1}^{L} \alpha_{i\ell}^2 \\
\text{s.t.} \quad & \sum_{k=1}^{N} y_{rk}^1 - s_r^+ = y_{r0}^1, \quad r \in R_1, \\
& \theta x_{i0}^1 - \sum_{k=1}^{N} \lambda_k x_{ik}^1 - s_i^- = 0, \quad i \in I_1, \\
& \left.\begin{aligned}
& \sum_{k=1}^{N} \lambda_k \gamma_{rk}(1) - \alpha_{r1}^1 = \gamma_{r0}(1), \\
& \sum_{k=1}^{N} \lambda_k \gamma_{rk}(2) + \alpha_{r1}^1 - \alpha_{r2}^1 = \gamma_{r0}(2), \\
& \vdots \\
& \sum_{k=1}^{N} \lambda_k \gamma_{rk}(L) + \alpha_{rL-1}^1 - \alpha_{rL}^1 = \gamma_{r0}(L),
\end{aligned}\right\} r \in R_2, \\
& \left.\begin{aligned}
& \delta_{i0}(L)\theta - \sum_{k=1}^{N} \lambda_k \delta_{ik}(L) - \alpha_{iL}^2 = 0, \\
& \delta_{i0}(L-1)\theta - \sum_{k=1}^{N} \lambda_k \delta_{ik}(L-1) + \alpha_{iL}^2 - \alpha_{iL-1}^2 = 0, \\
& \vdots \\
& \delta_{i0}(1)\theta - \sum_{k=1}^{N} \lambda_k \delta_{ik}(1) + \alpha_{i2}^2 - \alpha_{i1}^2 = 0,
\end{aligned}\right\} i \in I_2, \\
& \sum_{k=1}^{N} \lambda_k = 1, \\
& \lambda_k, s_r^+, s_i^-, \alpha_{r\ell}^1, \alpha_{i\ell}^2 \geqslant 0, \quad \theta \text{ unrestricted}.
\end{aligned}
\tag{3.5$'$}
$$

Here, we use $\{\lambda_k\}$ as the standard dual variables associated with the N ratio constraints, and the variables $\{\alpha^2_{i\ell}, \alpha^1_{r\ell}\}$ are the dual variables associated with the rank order constraints defined by Ψ. The slack variables s^+_r, s^-_i correspond to the lower bound restrictions on μ^1_r, υ^1_i.

Now, perform simple row operations on (3.5′) by replacing the ℓth constraint by the sum of the first ℓ constraints. That is, the second constraint (for those $r \in R_2$ and $i \in I_2$) is replaced by the sum of the first two constraints, constraint 3 by the sum of the first three, and so on. Letting

$$\overline{\gamma}_{rk}(\ell) = \sum_{n=1}^{l} \gamma_{rk}\gamma_{rk}(1) + \gamma_{rk}(2) + \cdots + \gamma_{rk}(\ell)$$

and

$$\overline{\delta}_{ik}(\ell) = \sum_{n=\ell}^{L} \delta_{ik}\delta_{ik}(L) + \delta_{ik}(L-1) + \cdots + \delta_{ik}(\ell),$$

problem (3.5′) can be rewritten as

$$\begin{aligned}
\min \quad & \theta - \varepsilon\sum_{r\in R_1} s^+_r - \varepsilon\sum_{i\in I_1} s^-_i \varepsilon^2 \sum_{r\in R_2}\sum_{l=1}^{L}\alpha^1_{rl} - \varepsilon^2\sum_{i\in I_2}\sum_{l=1}^{L}\alpha^2_{il} \\
\text{s.t.} \quad & \sum_{k=1}^{N}\lambda_k y^1_{rk} - s^+_r = y^1_{r0}, \quad r \in R_1, \\
& \theta x^1_{i0} - \sum_{k=1}^{N}\lambda_k x^1_{ik} - s^-_i = 0, \quad i \in I_1, \\
& \sum_{k=1}^{N}\lambda_k \overline{\gamma}_{rk}(\ell) - \alpha^1_{r\ell} = \overline{\gamma}_{r0}(\ell), \quad r \in R_2, \ell = 1,\ldots,L, \\
& \theta\overline{\delta}_{i0}(\ell) - \sum_{k=1}^{N}\lambda_k\overline{\delta}_{ik}(\ell) - \alpha^2_{il} = 0, \quad i \in I_2, \ell = 1,\ldots,L, \\
& \sum_{k=1}^{N}\lambda_k = 1, \\
& \lambda_k, s^+_r, s^-_i, \alpha^1_{rl}, \alpha^2_{il} \geqslant 0, \quad \text{all } i, r, \ell, k, \theta \text{ unrestricted in sign.}
\end{aligned} \tag{3.6′}$$

The dual of (3.6′) has the format:

$$\begin{aligned}
e_0 = \max \quad & \mu_0 + \sum_{r\in R_1}\mu^1_r y^1_{r0} + \sum_{r\in R_2}\sum_{\ell=1}^{L}\overline{w}^1_{r\ell}\overline{\gamma}_{r0}(\ell) \\
\text{s.t.} \quad & \sum_{i\in I_1}\upsilon^1_i x^1_{i0} + \sum_{i\in I_2}\sum_{\ell=1}^{L}\overline{w}^2_{i\ell}\overline{\delta}_{i0}(\ell) = 1, \\
& \mu_0 + \sum_{r\in R_1}\mu^1_r y_{rk} + \sum_{r\in R_2}\sum_{l=1}^{L}\overline{w}^1_{r\ell}\overline{\gamma}_{rk}(\ell) - \sum_{i\in I_1}\upsilon^1_i x^1_{ik} - \sum_{i\in I_2}\sum_{\ell=1}^{L}\overline{w}^2_{il}\overline{\delta}_{ik}(\ell) \leqslant 0, \quad \text{all } k, \\
& \mu^1_r, \upsilon^1_i \geqslant \varepsilon, \overline{w}^1_{r\ell}, \overline{w}^2_{i\ell} \geqslant \varepsilon^2,
\end{aligned} \tag{3.6}$$

which is a form of the VRS model. The slight difference between (3.6) and the conventional VRS model, is the presence of a different ε (i.e., ε^2) relating to the multipliers $w^1_{r\ell}$, $w^2_{i\ell}$, than is true for the multipliers μ^1_r, υ^1_i. It is observed that in (3.6′) the common L-value can easily be replaced by criteria specific values (e.g. L_r for output criterion r). The model structure remains the same, as does that of model (3.6). Of course, since the intention is to have an infinitesimal lower bound on multipliers (i.e., $\varepsilon = 0$), one can, from the start, restrict

$$\mu_r^1, \upsilon_i^1 \geqslant \varepsilon^2 \quad \text{and} \quad \mu_r^2, \upsilon_i^2 \geqslant \varepsilon.$$

This leads to a form of (3.6) where all multipliers have the same infinitesimal lower bounds, making (3.6) precisely a VRS model.

3.2. Criteria importance

The presence of ordinal data factors results in the need to *impute* values $y_r^2(\ell), x_i^2(\ell)$ to outputs and inputs, respectively, for DMUs that are ranked at positions on an L-point Likert or ordinal scale. Specifically, all DMUs ranked at that position will be credited with the same 'amount' $y_r^2(\ell)$ of output r ($r \in R_2$) and $x_i^2(\ell)$ of input i ($i \in I_2$).

A consequence of the change of variables undertaken above, to bring about linearization of the otherwise nonlinear terms, e.g., $w_{r\ell}^1 = \mu_r^2 y_r^2(\ell)$, is that at the optimum, all $\mu_r^2 = \varepsilon^2, \upsilon_i^2 = \varepsilon^2$. Thus, all of the ordinal criteria are relegated to the status of being of *equal importance*. Arguably, in many situations, one may wish to view the relative importance of these ordinal criteria (as captured by the μ_r^2, υ_i^2) in the same spirit as we have viewed the data values $\{y_{rk}^2\}$. That is, there may be sufficient information to be able to *rank* these criteria. Specifically, suppose that the R_2 output criteria can be grouped into L_1 categories and the I_2 input criteria into L_2 categories.

Now, replace the variables μ_r^2 by $\mu^2(m)$, and υ_i^2 by $\upsilon^2(n)$, and restrict:

$$\mu^2(m) - \mu^2(m+1) \geqslant \varepsilon, \quad m = 1, \ldots, L_1 - 1,$$

$$\mu^2(L_1) \geqslant \varepsilon,$$

and

$$\upsilon^2(n) - \upsilon^2(n+1) \geqslant \varepsilon, \quad n = 1, \ldots, L_2 - 1,$$

$$\upsilon^2(L_2) \geqslant \varepsilon.$$

Letting m_r denote the rank position occupied by output $r \in R_2$, and n_i the rank position occupied by input $i \in I_2$, we define the change of variables

$$w_{r\ell}^1 = \mu^2(m_r) y_r^2(\ell),$$

$$w_{i\ell}^2 = \upsilon^2(n_i) x_i^2(\ell).$$

The corresponding version of model (3.4) would see the lower bound restrictions μ_r^2, $\upsilon_i^2 \geqslant \varepsilon$ replaced by the above constraints on $\mu^2(m)$ and $\upsilon^2(n)$. Again, arguing that at the optimum in (3.4), these variables will be forced to their lowest levels, the resulting values of the $\mu^2(m)$, $\upsilon^2(n)$ will be

$$\mu^2(m) = (L_1 + 1 - m)\varepsilon, \quad \upsilon^2(n) = (L_2 + 1 - n)\varepsilon.$$

This implies that the lower bound restrictions on $w_{r\ell}^1, w_{i\ell}^2$ become

$$w_{r\ell}^1 \geqslant (L_1 + 1 - m_r)\varepsilon^2, \quad w_{i\ell}^2 \geqslant (L_2 + 1 - n_i)\varepsilon^2.$$

3.3. Solutions to applications

3.3.1. R&D project efficiency evaluation

When model (3.6′) is applied to the data of Table 1, the efficiency scores obtained are as shown in Table 3.

Table 3
Efficiency scores (non-ranked criteria)

Project	1	2	3	4	5	6	7	8	9	10
Score	0.76	0.73	1.00	0.67	1.00	0.82	0.67	0.67	0.55	0.37

Table 4
Efficiency scores (ranked criteria)

Project	1	2	3	4	5	6	7	8	9	10
Score	0.71	0.72	1.00	0.60	1.00	0.80	0.62	0.63	0.50	0.35

Here, projects 3 and 5 turn out to be 'efficient', while all other projects are rated well below 100%. In this particular analysis, ε was chosen as 0.03. In another run (not shown here) where $\varepsilon = 0.01$ was used, projects 3, 5 and 6 received ratings of 1.00, while all others obtained somewhat higher scores than those shown in Table 3. When a very small value of ε ($\varepsilon = 0.001$) was used, all except one of the projects was rated as efficient.

Clearly this example demonstrates the same degree of dependence on the choice of ε as is true in the standard DEA model, see Ali and Seiford (1993).

From the data in Table 1 it might appear that only project 3 should be efficient since 3 dominates project 5 in all factors except for the second input where project 3 rates fourth while project 5 rates fifth. As is characteristic of the standard ratio DEA model, a single factor can produce such an outcome. In the present case this situation occurs because $w_{25}^2 = 0.03$ while $w_{24}^2 = 0.51$. Consequently, project 5 is accorded an 'efficient' status by permitting the gap between w_{24}^2 and w_{25}^2 to be (perhaps unfairly) very large. Actually, the set of multipliers which render project 5 efficient also constitute an optimal solution for project 3.

If we further constrain the model by implementing criteria importance conditions as defined in the previous section, the relative positioning of the projects changes as shown in Table 4.

Hence, criteria importance restrictions can have an impact on the efficiency status of the projects.

3.4. Evaluation of telephone office efficiency

The data of Table 2 has been evaluated using Model (3.6′). We note again that both ordinal and numerical data are present. Both CRS and VRS models were applied, the results of which are presented in Table 5. Initially, in applying DEA in this application, no attempt was made to impose constraints on multipliers. Under the CRS structure, approximately half of the offices (17 of the 33) are declared efficient. With the VRS model, the number of efficient units climbs to 25 out of 33. When criteria importance is introduced, the efficiency status (efficient versus inefficient) changes for some units. As well, the relative sizes of efficiency scores change. Note, for example, that the relative positions of offices 10 and 11 are reversed under the constrained VRS model versus those assumed in the unconstrained model. As well, only 15 of the offices (rather than 25) are rated as being efficient.

Two very interesting phenomena characterize DEA problems containing ordinal data. If one examines in detail the outputs from the analysis of the example data, two observations can be made. First, it is the case that $\theta = 1$ for each project (whether efficient or inefficient). This means that each project is either on the frontier proper or an extension of the frontier. Second, if one were to use the CRS rather than VRS model, it would be observed that $\sum_k \lambda_k = 1$ for each project. The implication would seem to be that the two models (CRS and VRS) are equivalent in the presence of ordinal data. Moreover, since $\theta = 1$ in all cases, these models are as well equivalent to the additive model of Charnes et al. (1985). The following two theorems prove these results for the general case.

Table 5
Efficiency scores

DMU no.	CRS score	VRS score	VRS score-constrained
1	1	1	1
2	1	1	1
3	1	1	1
4	0.927	1	0.973
5	1	1	0.921
6	0.907	0.994	0.906
7	0.848	0.849	0.823
8	0.668	0.670	0.644
9	0.848	0.970	0.885
10	0.617	0.747	0.731
11	0.763	0.815	0.716
12	1	1	0.915
13	1	1	1
14	1	1	1
15	1	1	1
16	1	1	0.886
17	0.898	1	1
18	0.928	1	0.935
19	0.993	0.993	0.961
20	1	1	1
21	1	1	1
22	1	1	1
23	0.846	1	1
24	0.918	1	0.904
25	1	1	1
26	1	1	0.955
27	0.824	0.937	0.926
28	0.954	1	0.919
29	0.949	1	1
30	1	1	1
31	1	1	0.907
32	1	1	1
33	0.962	1	1

Theorem 3.3. *In problem* (3.6′), *if* I_2 *is non-empty*, $\theta = 1$ *at the optimum.*

Proof. By definition $\overline{\delta}_{ik}(\ell) = \sum_{n=\ell}^{L} \delta_{ik}(n)$. Thus $\overline{\delta}_{ik}(1) = 1$ for all k, and for any ordinal input i. From the constraint set of (3.6′), if $\sum_{k=1}^{N} \lambda_k = 1$, then since $\theta\overline{\delta}_{i0}(1) \geqslant \sum_{k=1}^{N} \lambda_k \overline{\delta}_{ik}(1)$ and since $\sum_{k=1}^{N} \lambda_k \overline{\delta}_{ik}(1) = 1$ (given that all members of $\{\overline{\delta}_{ik}(1)\}_{k=1}^{N}$ equal 1), it follows that $\theta\overline{\delta}_{i0}(1) \geqslant 1$. But since $\overline{\delta}_{i0}(1) = 1$, then $\theta \geqslant 1$ meaning that at the optimum $\theta = 1$. □

This rather unusual property of the DEA model in the presence of ordinal data is generally explainable by observing the dual form (3.5). It is noted that ε^2 plays the role of discriminating between the levels of relative importance of consecutive rank positions. If in the extreme case $\varepsilon = 0$, then any one rank position becomes as important as any other. This means that regardless of the rank position occupied by a DMU '0', that position can be credited with a higher weight, than those positions assumed by the peers of that DMU. Hence, every DMU will be deemed technically efficient. It is only the presence of positive gaps (defined by ε^2) between rank positions that renders a DMU inefficient via the slacks.

Theorem 3.4. *If R_2 and I_2 are both non-empty, then in the CRS version of problem (3.6′), $\sum_{k=1}^{N}\lambda_k = 1$ at the optimum.*

Proof. Reconsider problem (3.6′), but with the constraint $\sum_{k=1}^{N}\lambda_k = 1$ removed. As in Theorem 3.3, the constraint

$$\theta\overline{\delta}_{i0}(1) = \overline{\delta}_{i0}(1) \geqslant \sum_{k=1}^{N}\lambda_k\overline{\delta}_{ik}(1)$$

holds. But, since $\overline{\delta}_{ik}(1) = 1$ for all k, then it is the case that $\sum_{k=1}^{N}\lambda_k \leqslant 1$.

On the output side, however, $\overline{\gamma}_{rk}(L) = 1$ for all k, and any ordinal output r. But, since

$$\sum_{k=1}^{N}\lambda_k\overline{\gamma}_{rk}(L) \geqslant \overline{\gamma}_{r0}(L),$$

it follows that $\sum_{k=1}^{N}\lambda_k \geqslant 1$. Thus, $\sum_{k=1}^{N}\lambda_k = 1$. □

From Theorem 3.4, it follows that the VRS and CRS models are equivalent. Moreover, from Theorem 3.3, one may view these two models as equivalent to the additive model in that the objective function of (3.6′) is equivalent to maximizing the sum of all slacks.

It should be pointed out that the projection to the efficient frontier in model (3.6) treats the Likert scale $[1, L]$ as if it were a continuum, rather than as consisting of a set of discrete rank positions. Specifically, at the optimum in (3.6), any given projected value, e.g. $\theta\overline{\delta}_{i0}(\ell) - \alpha^2_{i\ell}$, $i \in I_2$, is not guaranteed to be one of the discrete points on the $[1, L]$ scale. For this reason, we refer to (3.6) as a *continuous projection* model. In this respect, the model can be viewed as providing a form of *upper bound* on the extent of reduction that can be anticipated in the ordinal inputs. Suppose, for example, that at the optimum $\theta\overline{\delta}_{i0}(\ell) - \alpha^2_{i\ell} = 2.7$, a *rank position* between the legitimate positions 2 and 3 on an L-point scale. Then, arguably, it is possible for $\overline{\delta}_{i0}(\ell)$ to be projected only to point 3, not further.

One can, of course, argue that the choosing of a specific number of rank positions L is generally motivated by an inability to be more discriminating (a larger L-value was not practical). At the proposed 'efficient' rank position of 2.7, we are claiming that the DMU '0' will be using more input i than a DMU ranked in 2nd place, but less input than one ranked in 3rd place. Thus, to some extent, this projection automatically has created an $L + 1$-point Likert scale, where previously the scale had contained only L points. The projection has permitted us to increase our degrees of discrimination.

In the special case treated by Cooper et al. (1999) where $L = N$, this issue never arises, as every DMU is entitled to occupy its own rank position on an N-point scale.

In Section 5 we will re-examine the discrete nature of the L-point scale and propose a model structure accordingly. In Section 4 we evaluate the IDEA concept of Cooper et al. (1999) in relation to the model developed above.

4. The continuous projection model and IDEA

Cooper et al. (1999) examine the DEA structure in the presence of *imprecise* data (IDEA) for certain factors. Zhu (2003a) and others have extended Cooper et al.'s (1999) earlier model. One particular form of imprecise data is a full ranking of the DMUs in an ordinal sense. Clearly, representation of rank data via a Likert scale, with L rank positions, is a generalization of the Cooper et al. (1999) structure wherein $L = N$.

To demonstrate this, we consider a full ranking of the DMUs in an ordinal sense and, to simplify the presentation, we suppose weak ordinal data can be expressed as (see Cooper et al. (1999); Zhu (2003a))

$$y_{r1} \leqslant y_{r2} \leqslant \cdots \leqslant y_{rk} \leqslant \cdots \leqslant y_{rn} \quad (r \in R_2), \tag{4.1}$$

$$x_{i1} \leqslant x_{i2} \leqslant \cdots \leqslant x_{ik} \leqslant \cdots \leqslant x_{in} \quad (i \in I_2). \tag{4.2}$$

When the set of Ψ is expressed as (4.1) and (4.2), model (3.5) can be expressed as

$$\begin{aligned}
e_0 = \max \quad & \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{r0}^1 + \sum_{r\in R_2} w_{r0}^1, \\
\text{s.t.} \quad & \sum_{i\in I_1} \upsilon_i^1 x_{i0}^1 + \sum_{i\in I_2} w_{i0}^2 = 1, \\
& \mu_0 + \sum_{r\in R_1} \mu_r^1 y_{rk}^1 + \sum_{r\in R_2} w_{rk}^1 - \sum_{i\in I_1} \upsilon_i^1 x_{ik}^1 - \sum_{i\in I_2} w_{ik}^2 \leqslant 0, \quad \text{all } k, \\
& - w_{r,k+1}^1 + w_{rk}^1 \leqslant 0, \quad k = 1,\ldots,N-1, \quad \text{all } r \in R_2, \\
& - w_{i,k+1}^2 + w_{ik}^2 \leqslant 0, \quad k = 1,\ldots,N-1, \quad \text{all } i \in I_2, \quad \mu_r^1, \upsilon_i^1 \geqslant \varepsilon, \quad r \in R_1, i \in I_1.
\end{aligned} \tag{4.3}$$

We next define $t_{rk}^1 = \upsilon_i^1 x_{ik}^1$ $(r \in R_1)$ and $t_{ik}^2 = \mu_r^1 y_{rk}^1$ $(i \in I_1)$. Then model (4.3) becomes the IDEA model of Cooper et al. (1999)[1]:

$$\begin{aligned}
e_0 = \max \quad & \mu_0 + \sum_{r\in R_1} t_{i0}^2 + \sum_{r\in R_2} w_{r0}^1 \\
\text{s.t.} \quad & \sum_{i\in I_1} t_{r0}^1 + \sum_{i\in I_2} w_{i0}^2 = 1, \\
& \mu_0 + \sum_{r\in R_1} t_{ik}^2 + \sum_{r\in R_2} w_{rk}^1 - \sum_{i\in I_1} t_{rk}^1 - \sum_{i\in I_2} w_{ik}^2 \leqslant 0, \quad \text{all } k, \\
& - w_{r,k+1}^1 + w_{rk}^1 \leqslant 0, \quad k = 1,\ldots N-1, \quad \text{all } r \in R_2, \\
& - w_{i,k+1}^2 + w_{ik}^2 \leqslant 0, \quad k = 1,\ldots,N-1, \quad \text{all } i \in I_2, \\
& t_{ik}^2, t_{rk}^1 \geqslant \varepsilon, \quad r \in R_1, i \in I_1.
\end{aligned} \tag{4.4}$$

We should point out that in the original IDEA model of Cooper et al. (1999), scale transformations on the input and output data, i.e., $\hat{x}_{ij} = x_{ij}/\max_j\{x_{ij}\}$, $\hat{y}_{rj} = y_{rj}/\max_j\{y_{rj}\}$ are done before new variables are introduced to convert the non-linear DEA model with ordinal data into a linear program. However, as demonstrated in Zhu (2003a), such scale transformations are unnecessary and redundant. As a result, the same variable alteration technique is used in Cook et al. (1993, 1996) and Cooper et al. (1999) in converting the non-linear IDEA model into linear programs. The difference lies in the fact that Cook et al. (1993, 1996) aims at converting the non-linear IDEA model into a conventional DEA model. To use the conventional DEA model based upon Cooper et al. (1999), one has to obtain a set of exact data from the imprecise or ordinal data (see Zhu, 2003a).

Based upon the above discussion, we know that the equivalence between the model of Section 3 and the IDEA model of Cooper et al. (1999) holds for any L if rank data are under consideration.

We finally discuss the treatment of strong versus weak ordinal relations in the model of Section 3.

Note that Ψ in Section 3 actually represents strong ordinal relations.[2] Cook et al. (1996) points out that efficiency scores can depend on ε in set Ψ and propose a model to determine a proper ε. Zhu (2003b) shows an

[1] The original IDEA model of Cooper et al. (1999) is discussed under the model of Charnes et al. (1978).

[2] As shown in Zhu (2003a), the expression in Ψ itself does not distinguish strong from weak ordinal relations if the IDEA model of Cooper et al. (1999, 2002) is used. Zhu (2003b) proposes a correct way to impose strong ordinal relations in the IDEA model of Cooper et al. (1999, 2002).

alternative approach in determining ε. Further, as shown in Zhu (2003b), part of weak ordinal relations can be replaced by strong ones without affecting the efficiency ratings and without the need for selecting the ε.

Alternatively, we can impose strong ordinal relations as $y_{rk} \geqslant \eta y_{r,k-1}$ and $x_{ik} \geqslant \eta x_{i,k-1}$ ($\eta > 1$).

5. Discrete projection for Likert scale data: An additive model

The model of the previous sections can be considered as providing a *lower bound* on the efficiency rating of any DMU. Arguably, as discussed above, projections may be infeasible in a strict ordinal ranking sense. The DEA structure explicitly implies that points on the frontier constitute permissible targets. Formalizing the R&D example of the Section 3, suppose that at two efficient (frontier) points k_1, k_2, it is the case that for an ordinal input $i \in I_2$, the respective rank positions are $\delta_{ik_1}(2) = 1$ and $\delta_{ik_2}(3) = 1$. That is, DMU k_1 is ranked in 2nd place on input *i*, while k_2 is ranked at 3rd place. Since all points on the line (facet) joining these two frontier units are to be considered as allowable projection points, then any 'rank position' between a rank of 2 and a rank of 3 is allowed. The DEA structure thus treats the rank range [1, L] as continuous, not discrete. In a 'full' rank order sense, one might interpret the projected point as giving DMU '0' a ranking just one position worse than that of DMU k_1, and thereby displacing k_2 and giving it (k_2) a rank that is one position worse than it had prior to the projection. This would mean that all DMUs ranked at or worse than is true for DMU k_2 would also be so displaced.

If DMUs are not rank ordered in the aforementioned sense, but rather are assigned to L (e.g. $L = 5$) rank positions, the described displacement does not occur. Specifically, if the *i*th ordinal input for DMU '0' is ranked in position ℓ_{ik} prior to projection toward the frontier, the only permissible other positions to which it can move are the discrete points $\ell_{ik} - 1$, $\ell_{ik} - 2$, ..., 1. The modeling of such discrete projections cannot, however, be directly accomplished within the radial framework, where each data value (e.g. $\overline{\delta}_{i_0}(\ell)$) is to be reduced by the same proportion $1 - \theta$.

The requirement to select from among a discrete set of rank positions (e.g. $\ell_{i_k} - 1, \ell_{i_k} - 2, \ldots, 1$) can be achieved from a form of the additive model as originally presented by Charnes et al. (1985). As discussed in Section (3), however, the VRS (and CRS) model is equivalent to the additive model. Thus, there is no loss of generality. An integer additive model version of (3.6′) can be expressed as follows: (We adopt here an 'invariant' form of the model, by scaling the objective function coefficients by the original data values.)

$$\max \quad \sum_{r\in R_1}(s_r^+/y_{r0}^1) + \sum_{i\in I_1}(s_i^-/x_{i0}^1) + \sum_{r\in R_2}\left(\sum_{\ell}\alpha_{r\ell}^1 \Big/ \sum_{\ell}\overline{\gamma}_{r_0}(\ell)\right) + \sum_{i\in I_2}\left(\sum_{\ell}\alpha_{i\ell}^2 \Big/ \sum_{\ell}\overline{\delta}_{i0}(\ell)\right) \tag{5.1a}$$

$$\text{s.t.} \quad \sum_{k=1}^{N}\lambda_k y_{rk}^1 - s_r^+ = y_{r0}^1, \quad r \in R_1, \tag{5.1b}$$

$$\sum_{k=1}^{N}\lambda_k x_{ik}^1 + s_i^- = x_{i0}^1, \quad i \in I_1, \tag{5.1c}$$

$$\sum_{k=1}^{N}\lambda_k \overline{\gamma}_{rk}\alpha_{r\ell}^1 \geqslant \overline{\gamma}_{r0}(\ell), \quad r \in R_2, \ell = 1,\ldots,L, \tag{5.1d}$$

$$\sum_{k=1}^{N}\lambda_k \overline{\delta}_{ik}\alpha_{i\ell}^2 \leqslant \overline{\delta}_{i0}(\ell), \quad i \in I_2, \ell = 1,\ldots,L, \tag{5.1e}$$

$$\sum_{k=1}^{N}\lambda_k = 1, \tag{5.1f}$$

$$\lambda_k \geqslant 0, \quad \text{all } k, s_i^-, s_r^+ \geqslant 0, i \in I_1, r \in R_1, \alpha_{r\ell}^1, \alpha_{r\ell}^2 \text{ integer}, r \in R_2, i \in I_2. \tag{5.1g}$$

The imposition of the integer restrictions on the $\alpha^1_{r\ell}$, $\alpha^2_{r\ell}$ is intended to create projections for inputs in I_2 and outputs in R_2 to points on the Likert scale.

Theorem 5.1. *The projections resulting from model (5.1) correspond to points on the Likert scale* $[1, L]$ *for inputs* $i \in I_2$ *and outputs* $r \in R_2$.

Proof. If for any $i \in I_2$, a DMU k is ranked at position ℓ_{ik}, then by definition

$$\overline{\delta}_{ik}(\ell) = \begin{cases} 0, & \ell > \ell_{ik}, \\ 1, & \ell \leqslant \ell_{ik}. \end{cases}$$

Similarly, for $r \in R_2$, if k is ranked at position ℓ_{rk}, then

$$\overline{\gamma}_{rk}(\ell) = \begin{cases} 0, & \ell < \ell_{rk}, \\ 1, & \ell \geqslant \ell_{rk}. \end{cases}$$

At the optimum of (5.1) (let $\{\hat{\lambda}_k\}$ denote the optimal λ_k), the $\{\sum_k \hat{\lambda}_k \overline{\delta}_{ik}(\ell)\}^L_{\ell=1}$ form a non-increasing sequence, i.e. $\sum^n_{k=1} \hat{\lambda}_k \overline{\delta}_{ik} \sum^n_{k=1} \hat{\lambda}_k \overline{\delta}_{ik}(\ell+1), \ell = 1, \ldots, L-1$. Similarly, the $\{\sum_k \hat{\lambda}_k \overline{\gamma}_{rk}(\ell)\}^L_{\ell=1}$ for a non-decreasing sequence. In constraint (5.1e), we let $\ell_{i\lambda(i)}$ denote the value of ℓ such that $\sum_k \hat{\lambda}_k \overline{\delta}_{ik} \ell_{i\lambda(i)}$, and $\sum_k \lambda_k \overline{\delta}_{ik}(\ell) > 0$ for $\ell < \ell_{i\lambda(i)}$. Clearly $\ell_{i\lambda(i)} \leqslant \ell_{io}$.

As well, at the optimum

$$\alpha^2_{i\ell} = \begin{cases} 0 & \text{for } \ell > \ell_{i_0}, \\ 1 & \text{for } \ell_{i_{\lambda_{(i)}}}, \\ 0 & \text{for } \ell < \ell_{i_{\lambda_{(i)}}}. \end{cases}$$

Hence, if we define the 'revised' $\overline{\gamma}$ and $\overline{\delta}$-values, by $\hat{\gamma}_{r0}(\ell) = \overline{\gamma}_{r0}(\ell) + \alpha^1_{r\ell}$ and $\hat{\delta}_{i0}(\ell) = \overline{\delta}_{i0}(\ell) - \alpha^2_{i\ell}$, then these define a proper rank position for input i (output (r)) of DMU '0'. That is,

$$\hat{\delta}_{i0}(\ell) = \begin{cases} 0, & \ell > \ell_{i\lambda(i)}, \\ 1, & \ell \leqslant \ell_{i\lambda(i)}, \end{cases}$$

and

$$\hat{\gamma}_{r0}\{\ell) = \begin{cases} 1, & \ell \geqslant \ell_{r\lambda(r)}, \\ 0, & \ell < \ell_{r\lambda(r)}, \end{cases}$$

meaning that the projected rank position of DMU '0' on e.g. input i is $\ell_{i\lambda(i)}$. $\square$

Unlike the radial models of Charnes et al. (1978) and Banker et al. (1984), the additive model does not have an associated convenient (or at least universally accepted) measure of efficiency. The objective function of (5.1) is clearly a combination of input and output projections. While various 'slacks based' measures have been presented in the literature (see e.g. Cooper et al., 2000), we propose a variant of the 'Russell Measure' as discussed in Fare and Lovell (1978). Specifically, define

$$\theta^1_i = 1 - s^-_i / x^1_{i0}, \ \ i \in 1; \quad \theta^2_i = 1 - \sum^L_{\ell=1} \alpha^2_{i\ell} \bigg/ \sum^L_{\ell=1} \overline{\delta}_{i0}(\ell), \ \ i \in I_2,$$

$$\phi^1_r = 1 + s^+_r / y^1_{r0}, \ \ r \in R_1; \quad \phi^2_r = 1 + \sum^L_{\ell=1} \alpha^1_{r\ell} \bigg/ \sum^L_{\ell=1} \overline{\gamma}_{r0}(\ell), \ \ r \in R_2.$$

Table 6
Efficiency scores β_1

Project no.	β_1	Peers
1	0.73	3
2	0.61	3
3	1.00	3
4	0.53	3
5	1.00	5
6	0.75	3
7	0.57	3
8	0.52	3
9	0.48	3
10	0.30	3

We define the efficiency measure of DMU '0' to be

$$\beta_1 = \left[\sum_{i\in I_1}\theta_i^1 + \sum_{i\in I_2}\theta_i^2 + \sum_{r\in R_1}(1/\phi_r^1) + \sum_{r\in R_2}(1/\phi_r^2)\right] \Big/ [|I_1| + |I_2| + |R_1| + |R_2|], \tag{5.2}$$

where $|I_1|$ denotes the cardinality of $I_1, \ldots$

The following property follows immediately from the definitions of θ_i^1, θ_i^2, ϕ_r^1, ϕ_r^2.

Property 5.1. *The efficiency measure* β_1 *satisfies the condition* $0 \leqslant \beta_1 \leqslant 1$.

It is noted that $\beta_1 = 1$ only in the circumstance that the DMU '0' is actually *on* the frontier. This means, of course that if a DMU is not on the frontier, but at the optimum of (5.1) has all $\alpha_{i\ell}^1$, $\alpha_{i\ell}^2 = 0$, it will be declared inefficient even though it is impossible for it to improve its position on the ordinal (Likert) scale. An additional and useful measure of *ordinal* efficiency is

$$\beta_2 = \left[\sum_{i\in I_2}\theta_i^2 + \sum_{r\in R_2}(1/\phi_r^2)\right] \Big/ [|I_1| + |R_2|]. \tag{5.3}$$

In this case $\beta_2 = 1$ for those DMUs for which further movement (improvement) along ordinal dimensions is not possible.

5.1. R&D example continued

Continuing the R&D example discussed in Section 3, we apply model (5.1) with the requisite requirement that only Likert scale projections are permitted in regard to ordinal factors. Table 6 presents the results.

As in the analysis of Section 3, all projects except for #3, #5 are inefficient. Interestingly, their *relative* sizes (in a rank order sense) agree with the outcomes shown in Table 3 of Section 3.

6. Conclusions

This paper has examined the use of ordinal data in DEA. Two general models are developed, namely, continuous and discrete projection models. The former aims to generate the maximum reduction in inputs (input-oriented model), without attention to the feasibility of the resulting projections in a Likert scale sense. The latter model specifically addresses the need to project to discrete points for ordinal factors.

We prove that in the presence of ordinal factors, CRS and VRS models are equivalent. As well, it is shown that in a pure technical efficiency sense, $\theta = 1$. Thus, it is only the slacks that render a DMU inefficient in regard to ordinal factors. The latter also implies that projections in the VRS (and CRS) sense are the same as those arising from the additive model. This provides a rationale for reverting to the additive model to facilitate projection in the discrete (versus continuous) model described in Section 5.

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