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Theory and Methodology

Super-efficiency and DEA sensitivity analysis

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Abstract

This paper discusses and reviews the use of super-efficiency approach in data envelopment analysis (DEA) sensitivity analyses. It is shown that super-efficiency score can be decomposed into two data perturbation components of a particular test frontier decision making unit (DMU) and the remaining DMUs. As a result, DEA sensitivity analysis can be done in (1) a general situation where data for a test DMU and data for the remaining DMUs are allowed to vary simultaneously and unequally and (2) the worst-case scenario where the efficiency of the test DMU is deteriorating while the efficiencies of the other DMUs are improving. The sensitivity analysis approach developed in this paper can be applied to DMUs on the entire frontier and to all basic DEA models. Necessary and sufficient conditions for preserving a DMU's efficiency classification are developed when various data changes are applied to all DMUs. Possible infeasibility of super-efficiency DEA models is only associated with extreme-efficient DMUs and indicates efficiency stability to data perturbations in all DMUs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

During the recent years, the issue of sensitivity and stability of data envelopment analysis (DEA) results has been extensively studied. Some studies focus on the sensitivity of DEA results to the variable and model selection, e.g., Ahn and Seiford (1993) and Smith (1997). Most of the DEA sensitivity analysis studies focus on the misspecification of efficiency classification of a test decision making unit (DMU). However, we shall note that DEA is an extremal method in the sense that all extreme points are characterized as efficient. If data entry errors occur for various DMUs, the resulting isoquant may vary substantially (Sexton et al., 1986). In the current paper, as in many other DEA sensitivity studies, we say that the calculated frontiers of DEA models are stable if the frontier DMUs that determine the DEA frontier remain on the frontier after the particular data perturbations are made for all DMUs.

By updating the inverse of the basis matrix associated with a specific efficient DMU in a DEA linear programming problem, Charnes et al. (1985a) study the sensitivity of DEA model to a single output change. This is followed by a series

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of sensitivity analysis articles by Charnes and Neralic in which sufficient conditions preserving efficiency are determined (Charnes and Neralic, 1990).

Another type of DEA sensitivity analysis is based on super-efficiency DEA approach in which a test DMU is not included in reference set (Andersen and Petersen, 1993; Seiford and Zhu, 1999). Charnes et al. (1992), Rousseau and Semple (1995) and Charnes et al. (1996) develop a super-efficiency DEA sensitivity analysis technique for the situation where simultaneous proportional change is assumed in all inputs and outputs for a specific DMU under consideration. This data variation condition is relaxed in Zhu (1996) and Seiford and Zhu (1998a) to a situation where inputs or outputs can be changed individually and the entire (largest) stability region which encompasses that of Charnes et al. (1992) is obtained. As a result, the condition for preserving efficiency of a test DMU is necessary and sufficient.

The DEA sensitivity analysis methods we have just reviewed are all developed for the situation where data variations are only applied to the test efficient DMU and the data for the remaining DMUs are assumed fixed. Obviously, this assumption may not be realistic, since possible data errors may occur in each DMU. Seiford and Zhu (1998b) generalize the technique in Zhu (1996) and Seiford and Zhu (1998a) to the worst-case scenario where the efficiency of the test DMU is deteriorating while the efficiencies of the other DMUs are improving. In their method, same maximum percentage data change of a test DMU and the remaining DMUs is assumed and sufficient conditions for preserving an extreme-efficient DMU's efficiency are determined. Note that Thompson et al. (1994) use the strong complementary slackness condition (SCSC) multipliers to analyze the stability of the CCR model (Charnes et al., 1978) when the data for all efficient and all inefficient DMUs are simultaneously changed in opposite directions and in same percentages. Although the data variation condition is more restrictive in Seiford and Zhu (1998b) than that in Thompson et al. (1994), the super-efficiency-based approach may generate a larger stability region than the SCSC method does. Also, the SCSC method is dependent on a particular SCSC solution, among others, and therefore the resulting analysis may vary.

In this paper, we focus on the DEA sensitivity analysis methods based on super-efficiency DEA models. For the DEA sensitivity analysis based on the inverse of basis matrix, the reader is referred to Neralic (1994). It is well known that certain super-efficiency DEA models may be infeasible for some extreme-efficient DMUs. Seiford and Zhu (1999) develop the necessary and sufficient conditions for infeasibility of various super-efficiency DEA models. Although the super-efficiency DEA models employed in Charnes et al. (1992) and Charnes et al. (1996) do not encounter the infeasibility problem, the models used in Seiford and Zhu (1998a) do. Seiford and Zhu (1998a) discover the relationship between infeasibility and stability of efficiency classification. That is, infeasibility means that the CCR efficiency of the test DMU remains stable to data changes in the test DMU. Furthermore, Seiford and Zhu (1998b) show that this relationship is also true for the simultaneous data change case and other DEA models, such as BCC model of Banker et al. (1984) and additive model of Charnes et al. (1985b). This finding is critical since super-efficiency DEA models in Seiford and Zhu (1998b) are frequently infeasible for real-world data sets, indicating efficiency stability with respective to data variations in inputs/outputs associated with infeasibility.

This study attempts to generalize the results in Seiford and Zhu (1998b) to a situation where variable percentage data changes are assumed for a test DMU and for the remaining DMUs. We consider the same worst-case analysis as in Seiford and Zhu (1998b). It is shown that a particular super-efficiency score can be decomposed into two data perturbation components of a particular test DMU and the remaining DMUs. Also, necessary and sufficient conditions for preserving a DMU's efficiency classification are developed when various data changes are applied to all DMUs. As a result, DEA sensitivity analysis can be easily applied if we employ various super-efficiency DEA models.

The current article proceeds as follows. In Section 2, we present some basic DEA models and define the data variations in a test frontier DMU

and the remaining DMUs. In Section 3, we develop our sensitivity analysis method for simultaneous data changes in all DMUs. The sensitivity of the DEA constant returns to scale (CRS) model (e.g., CCR model) and of the DEA VRS (variable returns to scale) models (e.g., BCC model) is discussed. The assumption of same percentage data change of all DMUs in Seiford and Zhu (1998b) is relaxed. We discuss the efficiency stability of all frontier DMUs. Conclusions are given in Section 4.

2. Preliminaries

Suppose we have a set of n DMUs. Each DMU_j $(j = 1, \ldots, n)$, produces s different outputs y_{rj} $(r = 1, \ldots, s)$ utilizing m different inputs x_{ij} $(i = 1, \ldots, m)$. Two types of orientation DEA models are often used to evaluate DMUs' relative efficiency: CRS models, such as CCR model, and VRS models, such as BCC model. For example, an input-oriented CRS model can be expressed as

$$\begin{aligned}
&\theta^{0^*}_{\mathrm{CRS}} = \min\ \theta^{0}_{\mathrm{CRS}}\\
&\text{s.t.}\quad \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = \theta^{o}_{\mathrm{CRS}} x_{i0},\\
&\qquad i = 1, 2, \ldots, m,\\
&\qquad \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{r0},\\
&\qquad r = 1, 2, \ldots, s,\\
&\qquad \theta^{0}_{\mathrm{CRS}},\ \lambda_j, s_i^-, s_r^+ \geqslant 0,
\end{aligned} \tag{1}$$

where, x_{i0} and y_{r0} are respectively the ith input and rth output for DMU_0 under evaluation. If we add an additional convex constraint of $\sum_{j=1}^{n}\lambda_j = 1$ into (1), we obtain an input-oriented VRS model.

The dual to (1) can be written as

$$\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{r0}\\
&\text{s.t.}\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, \ldots, n,\\
&\qquad \sum_{i=1}^{m} v_i x_{i0} = 1,\\
&\qquad u_r, v_i \geqslant 0.
\end{aligned} \tag{2}$$

Based on Charnes et al. (1991), a set of DMUs can be partitioned into two groups: frontier DMUs and non-frontier DMUs. Furthermore, the frontier DMUs consist of DMUs in set E (extreme efficient), set E′ (efficient but not an extreme point) and set F (weakly efficient or frontier point but with non-zero slacks). If $\theta^{0^*}_{\mathrm{CRS}} = 1$ then DMU_0 is a CRS frontier point. We may use a super-efficiency DEA model to identify the classification of DMU_0. That is,

$$\begin{aligned}
&\theta_0^{\text{sup } er^*} = \min\ \theta_0^{\text{sup } er}\\
&\text{s.t.}\quad \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant \theta_0^{\text{sup } er} x_{i0},\\
&\qquad i = 1, 2, \ldots, m,\\
&\qquad \sum_{j=1,\ j\neq 0}^{n} \lambda_j y_{rj} \geqslant y_{r0},\\
&\qquad r = 1, 2, \ldots, s,\\
&\qquad \theta_0^{\text{sup } er},\ \lambda_j\ (j \neq 0) \geqslant 0.
\end{aligned} \tag{3}$$

Based on Thrall (1996, p. 117), we have:

(a) $\theta_0^{\text{sup } er} > 1$ or (3) is infeasible if and only if $\mathrm{DMU}_0 \in$ set E, and

(b) $\theta_0^{\text{sup } er} = 1$ if and only if $\mathrm{DMU}_0 \in$ set $\mathrm{E}' \cup \mathrm{F}$.

In (b), if non-zero input/output slack values are detected in (1), then $\mathrm{DMU}_0 \in$ set F. Consider the four DMUs, A, B, C and D with two-input and one single output in Table 1. Using model (3) yields that A, B and C belong to set E and D belongs to set $\mathrm{E}' \cup \mathrm{F}$. Furthermore, there is a non-zero slack value on its second input in (1), therefore D $\in$ set F.

Similarly, we can have an output-oriented CRS super-efficiency model. That is,

$$\begin{aligned}
&\phi_0^{\text{sup } er^*} = \max\ \phi_0^{\text{sup } er}\\
&\text{s.t.}\quad \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant \phi_0^{\text{sup } er} y_{r0},\\
&\qquad r = 1, \ldots, s,\\
&\qquad \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, \quad i = 1, \ldots, m,\\
&\qquad \phi_0^{\text{sup } er}, \lambda_j\ (j \neq 0) \geqslant 0.
\end{aligned} \tag{4}$$

Table 1
Sample DMUs

DMU	y	x_1	x_2	$\theta_0^{\text{sup } er^*} = \theta^{0^*}_{\mathbf{I}=\{1,2\}}$	$\theta^{0^*}_{\mathbf{I}=\{1\}}$	$\theta^{0^*}_{\mathbf{I}=\{2\}}$
A	1	2	5	15/13	5/4	7/5
B	1	3	3	26/21	14/9	17/12
C	1	6	2	3/2	Infeasible	3/2
D	1	2	7	1	1	5/7

For other super-efficiency DEA models, please refer to Seiford and Zhu (1999) where they provide a complete list of super-efficiency DEA models and develop the necessary and sufficient conditions for the infeasibility of various super-efficiency DEA models. As we shall shortly see, modifications to super-efficiency DEA models (3) and (4) have been used as a sensitivity analysis tool in DEA.

3. Super-efficiency and sensitivity analysis

The frontier points in DEA are of primary importance as they define the DEA frontier. The current paper will discuss the stability of efficiency classification for such DMUs. That is, we are interested in whether DMU_0 will still be a frontier point after data perturbations in all the DMUs.

Since an increase of any output or a decrease of any input cannot worsen the efficiency of DMU_0, we restrict our attention to decrease in outputs and increase in inputs for DMU_0. In order to simultaneously consider the data changes for other DMUs, as in Seiford and Zhu (1998b), we suppose increased output and decreased input for all other DMUs. That is, our discussion is based on a worst-case scenario in which efficiency of DMU_0 declines and the efficiencies of all other DMU_j $(j \neq 0)$ improve.

Let **I** and **O** denote, respectively, the input and output subsets in which we are interested. That is, we consider the data changes in set **I** and set **O**. Then the simultaneous data perturbations in input/output of all DMU_j $(j \neq 0)$ and DMU_0 can be written as *percentage data perturbation* (*variation*):

for DMU_0,

$$\begin{cases} \hat{x}_{i0} = \delta_i x_{i0}, & \delta_i \geqslant 1,\ i \in \mathbf{I}, \\ \hat{x}_{i0} = x_{i0}, & i \notin \mathbf{I}, \end{cases}$$

and

$$\begin{cases} \hat{y}_{r0} = \tau_r y_{r0}, & 0 < \tau_r \leqslant 1,\ r_r \in \mathbf{O}, \\ \hat{y}_{r0} = y_{r0}, & r \notin \mathbf{O}; \end{cases}$$

for DMU_j $(j \neq 0)$,

$$\begin{cases} \hat{x}_{ij} = x_{ij}/\tilde{\delta}_i, & \tilde{\delta}_i \geqslant 1,\ i \in \mathbf{I}, \\ \hat{x}_{ij} = x_{ij}, & i \notin \mathbf{I}, \end{cases}$$

and

$$\begin{cases} \hat{y}_{rj} = y_{rj}/\tilde{\tau}_r, & 0 < \tilde{\tau}_r \leqslant 1,\ r \in \mathbf{O}, \\ \hat{y}_{rj} = y_{rj}, & r \notin \mathbf{O}; \end{cases}$$

where $(\hat{})$ represents adjusted data. Note that the data perturbations represented by δ_i and $\tilde{\delta}_i$ (or τ_r and $\tilde{\tau}_r$) can be different for each $i \in \mathbf{I}$ (or $r \in \mathbf{O}$).

Before we turn to the next section, we provide the following lemma associated with DMUs in set F.

Lemma 1. *Suppose* $\text{DMU}_0 \in$ set F *with non-zero input/output slack values associated with* set **I**/set **O**. *Then* DMU_0 *with inputs of* $\hat{x}_{i0}$ *and outputs of* $\hat{y}_{r0}$ *as defined above still belongs to* set F *when other* DMUs *are fixed.*

Proof. Applying the complementary slackness theorem for models (1) and (2), we have $s_i^{-*} v_i^* = s_r^{+*} u_r^* = 0$. Since $s_i^{-*} \neq 0$ for $i \in \mathbf{I}$ and $s_r^{+*} \neq 0$ for $r \in \mathbf{O}$, we have $v_i^* = 0$ for $i \in I$ and $u_r^* = 0$ for $r \in \mathbf{O}$. Therefore, v_i^* and u_r^* is a feasible solution to (2) for DMU_0 with inputs of $\hat{x}_{i0}$ and outputs of $\hat{y}_{r0}$. Note that

$$\sum_{r=1}^{s} u_r^* \hat{y}_{r0} = \sum_{r \notin \mathbf{O}} u_r^* \hat{y}_{r0} = \sum_{r \notin \mathbf{O}} u_r^* y_{r0} = 1$$

indicating that the maximum value of 1 is achieved. Therefore, the DMU_0 still belongs to set F. □

3.1. Sensitivity analysis under CRS model

We first modify models (3) and (4) to the following two super-efficiency DEA models which are employed by Seiford and Zhu (1998b) to develop sensitivity analysis approach when the same percentage in data changes of DMU_0 and DMU_j $(j \neq 0)$ is assumed:

$$\begin{aligned}
&\theta_{\mathbf{I}}^{0^*} = \min \theta_{\mathbf{I}}^0 \\
&\text{s.t.} \quad \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant \theta_{\mathbf{I}}^0 x_{i0}, \quad i \in \mathbf{I}, \\
&\qquad \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, \quad i \notin \mathbf{I}, \\
&\qquad \sum_{j=1,\ j\neq 0}^{n} \lambda_j y_{rj} \geqslant y_{r0}, \quad r = 1, 2, \ldots, s, \\
&\qquad \theta_{\mathbf{I}}^0, \lambda_j\ (j \neq 0) \geqslant 0
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\phi_{\mathbf{O}}^{0^*} = \max \phi_{\mathbf{O}}^0 \\
&\text{s.t.} \quad \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant \phi_{\mathbf{O}}^0 y_{r0}, \quad r \in \mathbf{O}, \\
&\qquad \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj}\ \geqslant\ y_{r0}, \quad r \notin \mathbf{O}, \\
&\qquad \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ x_{ij}\ \leqslant\ x_{i0}, \quad i = 1, \ldots, m, \\
&\qquad \phi_{\mathbf{O}}^0, \lambda_j\ (j \neq 0) \geqslant\ 0.
\end{aligned} \tag{6}$$

If

$$\mathbf{I} = \{k\},\ k \in \{1, \ldots, m\} \text{ and } \mathbf{O} = \{l\},\ l \in \{1, \ldots, s\},$$

then optimal values of

$$\theta_{\mathbf{I}}^{0^*} = \left(\theta_k^{0^*}\right)\ (k = 1, \ldots, m) \text{ and } \phi_{\mathbf{O}}^{0^*} = \left(\phi_l^{0^*}\right)\ (l = 1, \ldots, s)$$

are used by Zhu (1996) and Seiford and Zhu (1998a) to compute the exact (largest) stability region for DMU_0 when variations in the data are only applied to DMU_0.

Models (5) and (6) measure the maximum increase rate of inputs associated with **I** and the maximum decrease rate of outputs associated with **O**, respectively, required for DMU_0 to reach the frontier of DMU_j $(j \neq 0)$ when other inputs and outputs are kept at their current levels. For example, consider B in Table 1 (Fig. 1) and model (5). If $\mathbf{I} = \{1\}$, $\theta_{\{1\}}^{B^*} = \frac{14}{9}$ indicates that B reaches B_1 by changing its x_1 to $\frac{14}{9} \times 3 = \frac{14}{3}$. If $\mathbf{I} = \{2\}$, $\theta_{\{2\}}^{B^*} = \frac{17}{12}$ indicates that B reaches B_2 by changing its x_2 to $\frac{17}{12} \times 3 = \frac{17}{4}$. If $\mathbf{I} = \{1, 2\}$, $\theta_{\{1,2\}}^{B^*} = \theta_{B}^{\text{sup } er^*} = \frac{26}{21}$ gives the input increase rate for B in order to reach B′.

Associated with the optimal values in models (3)–(6), we have:

Lemma 2.
(i) *If* $\theta_0^{\text{sup } er} = 1$, *then* $\theta_{\mathbf{I}}^{0^*} \leqslant 1$;
(ii) *If* $\phi_0^{\text{sup } er^*} = 1$, *then* $\phi_{\mathbf{O}}^{0^*} \geqslant 1$.

Proof. The proof is obvious from the fact that $\theta_0^{\text{sup } er} = 1$ is a feasible solution to (5) and $\phi_0^{\text{sup } er^*} = 1$ is a feasible solution to (6). □

Lemma 3.
(i) *If* $\theta_0^{\text{sup } er} = 1$ *and* $\theta_{\mathbf{I}}^{0^*} < 1$, *then* $DMU_0 \in F$;
(ii) *If* $\phi_0^{\text{sup } er^*} = 1$ *and* $\phi_{\mathbf{O}}^{0^*} > 1$, *then* $DMU_0 \in F$.

Proof.
(i) $\theta_0^{\text{sup } er} = 1$ indicates that $DMU_0 \in E' \cup F$. $\theta_{\mathbf{I}}^{0^*} < 1$ further indicates that there are non-zero slack values in x_{i0} for $i \in \mathbf{I}$. Thus, $DMU_0 \in F$.
(ii) The proof is similar to that of (i). □

Theorem 1.
(i) *If* $\theta_0^{\text{sup } er} = 1$ *and* $\theta_{\mathbf{I}}^{0^*} < 1$, *then for any* $\delta_i \geqslant 1$ *and* $\tilde{\delta}_i \geqslant 1$ $(i \in \mathbf{I})$, DMU_0 *remains in* set F;
(ii) *If* $\phi_0^{\text{sup } er^*} = 1$ *and* $\phi_{\mathbf{O}}^{0^*} > 1$, *then for any* $0 < \tau_r \leqslant 1$ *and* $0 < \tilde{\tau}_r \leqslant 1 (r \in \mathbf{O})$, DMU_0 *remains in* set F.

Proof.
(i) From Lemma 3, we know that $DMU_0 \in F$ with non-zero slack values in x_{i0} for $i \in \mathbf{I}$. Based on Lemma 1 and the proof of Lemma 1, we

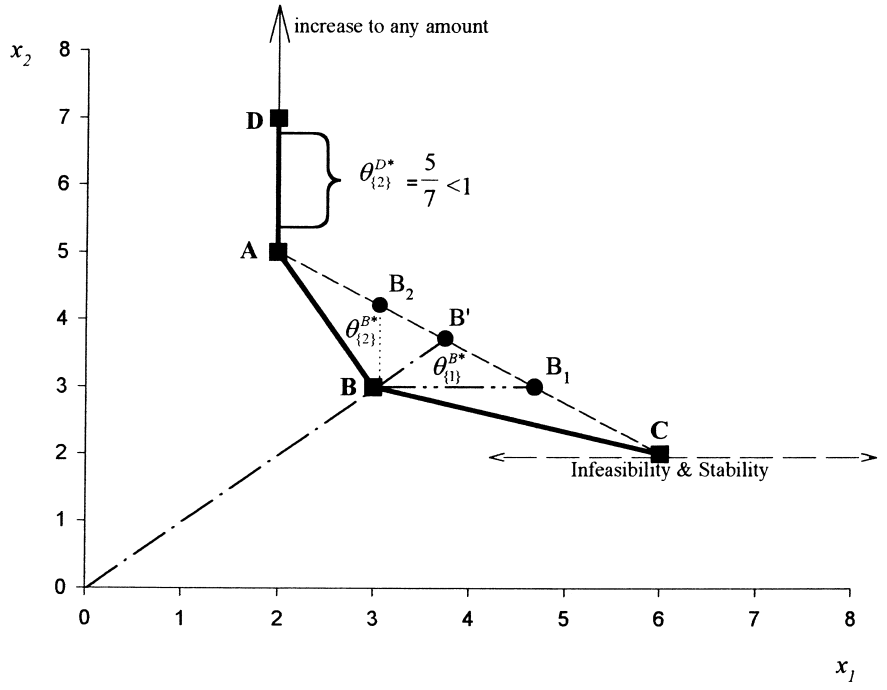

Fig. 1. Super-efficiency and sensitivity analysis.

know that for any $\delta_i \geqslant 1$ and $\tilde{\delta}_i \geqslant 1$, with an objective function value of 1, v_i^* and u_r^* is a feasible solution to (2) in which inputs are replaced by $\hat{x}_{ij}$ for $i \in \mathbf{I}$ and x_{ij} for $i \notin \mathbf{I}$. Thus, DMU_0 remains in set F after input data changes set $\mathbf{I}$ in all DMUs.

(ii) The proof is similar to that of (i). □

In fact, Lemma 1 and Theorem 1 indicate that the classification of DMUs in set F is stable under any data perturbations in all DMUs occurred in inputs (outputs) which have non-zero slack values in DMU_0. For example, if $\mathbf{I} = \{2\}$, then model (5) yields $\theta^{\mathrm{D}^*}_{\{2\}} = \frac{5}{7} < 1$ for D indicating that D has non-zero slack value in its second input. From Fig. 1, it is clear that D can increase its x_2 to any amount and still belongs to set F while other DMUs, A, B and C decrease their amount of x_2. This finding is very useful for the sensitivity analysis of the DMUs in set F.

Theorem 1 gives the sufficient condition for $\mathrm{DMU}_0 \in$ set F to preserve its efficiency classification. By Lemma 2, we immediately have:

Corollary 1.

(i) *If for any* $\delta_i \geqslant 1$ *and* $\tilde{\delta}_i \geqslant 1$ $(i \in \mathbf{I})$, DMU_0 *remains in* set F, *then* (a) $\theta_0^{\mathrm{sup}\ er} = 1$ *and* $\theta_{\mathbf{I}}^{0^*} < 1$, or (b) $\theta_0^{\mathrm{sup}\ er} = 1$ *and* $\theta_{\mathbf{I}}^{0^*} = 1$;

(ii) *If for any* $0 < \tau_r \leqslant 1$ *and* $0 < \tilde{\tau}_r \leqslant 1$ $(r \in \mathbf{O})$, DMU_0 *remains in* set F, *then* (a) $\phi_0^{\mathrm{sup}\ er^*} = 1$ *and* $\phi_{\mathbf{O}}^{0^*} > 1$, *or* (b) $\phi_0^{\mathrm{sup}\ er^*} = 1$ *and* $\phi_{\mathbf{O}}^{0^*} = 1$.

Corollary 1 implies that for $\mathrm{DMU}_0 \in$ set F, some inputs without slack values may also be increased while preserving the efficiency of DMU_0. For example, consider two DMUs: $\mathrm{DMU}_1 = (y, x_1, x_2, x_3)$ and $\mathrm{DMU}_2 = (y, x_1, x_2, \pi x_3)$, where $\pi > 1$, a constant. Obviously, $\mathrm{DMU}_1 \in$ set E and $\mathrm{DMU}_2 \in$ set F with non-zero slack value on the third input. Now, let $\mathbf{I} = \{2, 3\}$. We have that DMU_2 with $(y, x_1, \delta x_2, \delta \pi x_3)$ $(\delta > 1)$ remains in set F while DMU_1 is changed to $(y, x_1, x_2/\tilde{\delta}, x_3/\tilde{\delta}) \in$ set E $(\tilde{\delta} > 1)$. In this situation, $\theta^{2^*}_{\{2,3\}} = 1$ in (5).

From Lemma 2, we know that $\theta_{\mathbf{I}}^{0^*}$ or $\phi_{\mathbf{O}}^{0^*}$ may also be equal to one. Obviously, in this situation, $\mathrm{DMU}_0 \in$ set E′ or set F and our approach indicates that no data variations are allowed in DMU_0

and other DMUs. In fact, any data perturbation defined above will change the efficiency classification of DMUs in set E′. Note also that $\theta^{D^*}_{\{1\}} = 1$ for D in Table 1. Thus, any data variation in the first input will let D become non-frontier point (see Fig. 1).

As indicated by Seiford and Zhu (1998b), model (5) (or (6)) may be infeasible even when model (3) (or (4)) is feasible. Furthermore, from Lemma 2, we have:

Corollary 2. *Infeasibility of model* (5) *or model* (6) *can only be associated with extreme-efficient* DMU*s in* set E.

Proof. Lemma 2 implies that models (5) and (6) are always feasible for DMUs in set E′ or set F. Also, models (5) and (6) are always feasible for non-frontier DMUs. Therefore, infeasibility of models (5) and (6) may only occur for extreme-efficient DMUs in set E. □

Seiford and Zhu (1998b) show that infeasibility of a super-efficiency DEA model means stability of the efficiency classification of DMU_0 with respect to the changes of corresponding inputs and (or) outputs in all DMUs. We summarize Seiford and Zhu's (1998b) finding as the following theorem:

Theorem 2.

(i) *A specific super-efficiency DEA model associated with* set **I** *is infeasible, if and only if for any* $\delta_i \geqslant 1$ *and* $\tilde{\delta}_i \geqslant 1$ $(i \in \mathbf{I})$, DMU_0 *remains extreme-efficient.*

(ii) *A specific super-efficiency DEA model associated with* set **O** *is infeasible, if and only if for any* $0 < \tau_r \leqslant 1$ *and* $0 < \tilde{\tau}_r \leqslant 1$ $(r \in \mathbf{O})$, DMU_0 *remains extreme-efficient.*

Theorem 2 indicates that, for example, if model (5) is infeasible, then DMU_0 will still be extreme-efficient no matter how much its inputs associated with set **I** are increased while the corresponding inputs of other DMUs are decreased. Consider C in Table 1. If $\mathbf{I} = \{1\}$, then model (5) is infeasible. (Note that model (3) is feasible for C.) From Fig. 1, it is clear that C will remain extreme-efficient if its first input is increased to any amount while DMUs A, C and D decrease their amount of x_1.

In the discussion to follow, we assume that super-efficiency DEA models (5) and (6) are feasible. Otherwise, the efficiency classification of DMU_0 is stable to data perturbations in all DMUs by Theorem 2.

Lemma 4.

(i) *If model* (5) *is feasible and* $\theta_0^{\text{sup } er} > 1$ *then* $\theta_{\mathbf{I}}^{0^*} > 1$;

(ii) *If model* (6) *is feasible and* $\phi_0^{\text{sup } er^*} < 1$ *then* $\phi_{\mathbf{O}}^{0^*} < 1$.

Proof.

(i) Suppose $\theta_{\mathbf{I}}^{0^*} \leqslant 1$. Then the input constraints of (5) turn into

$$\begin{cases} \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant \theta_{\mathbf{I}}^{0^*} x_{i0} \leqslant x_{i0}, & i \in \mathbf{I}, \\ \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, & i \notin \mathbf{I}, \end{cases}$$

which indicates that $\theta_0^{\text{sup } er} = 1$ is a feasible solution to (3). Therefore, $\theta_0^{\text{sup } er^*} \leqslant 1$. A contradiction. Thus, $\theta_{\mathbf{I}}^{0^*} > 1$.

(ii) The proof is similar to that of (i). □

Lemma 4 indicates that if $DMU_0 \in$ set E and model (5) (or model (6)) is feasible, then $\theta_{\mathbf{I}}^{0^*}$ must be greater than one (or $\phi_{\mathbf{O}}^{0^*}$ must be less than one). We next study the efficiency stability of extreme efficient DMUs and we relax the assumption that same percentage change holds for data variation of DMU_0 and DMU_j $(j \neq 0)$ and generalize the results in Seiford and Zhu (1998b).

Theorem 3. *Suppose* $\theta_0^{\text{sup } er^*} > 1$ *and* $\phi_0^{\text{sup } er^*} < 1$, *then*

(i) *If* $1 \leqslant \delta_i \tilde{\delta}_i < \theta_{\mathbf{I}}^{0^*}$ *for* $i \in \mathbf{I}$, *then* DMU_0 *remains extreme-efficient. Furthermore, if equality holds for* $\delta_i \tilde{\delta}_i = \theta_{\mathbf{I}}^{0^*}$, *that is,* $1 \leqslant \delta_i \tilde{\delta}_i \leqslant \theta_{\mathbf{I}}^{0^*}$, *then* DMU_0 *remains on the frontier, where* $\theta_{\mathbf{I}}^{0^*}$ *is the optimal value to* (5).

(ii) *If* $\phi_{\mathbf{O}}^{0^*} < \tau_r \tilde{\tau}_r \leqslant 1$ *for* $r \in \mathbf{O}$, *then* DMU_0 *remains extreme-efficient. Furthermore, if equality holds for* $\tau_r \tilde{\tau}_r = \phi_{\mathbf{O}}^{0^*}$, that is, $\phi_{\mathbf{O}}^{0^*} \leqslant \tau_r \tilde{\tau}_r \leqslant 1$, *then*

DMU_0 *remains on the frontier, where* $\phi_{\mathbf{O}}^{0^*}$ *is the optimal value to* (6).

Proof.

(i) Note that from Lemma 4, $\theta_{\mathbf{I}}^{0^*} > 1$. Now suppose $1 \leqslant \delta_i^0 \quad \tilde{\delta}_i^0 < \theta_{\mathbf{I}}^{0^*}$, and DMU_0 is not extreme-efficient when $\hat{x}_{i0} = \delta_i^0 x_{i0}$ and $\hat{x}_{ij} = x_{ij}/\tilde{\delta}_i^0$, $\in \mathbf{I}$. Then, there exist $\lambda_j \ (j \neq 0) \geqslant 0$ and $\theta_0^{\sup\ er^*} \leqslant 1$ in (3) such that

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j \frac{x_{ij}}{\tilde{\delta}_i^0} \leqslant \theta_0^{\sup\ er^*} \delta_i^o x_{i0}, \quad i \in \mathbf{I},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant \theta_0^{\sup\ er^*} x_{i0} \leqslant x_{i0}, \quad i \notin \mathbf{I},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j y_{rj} \geqslant y_{r0}, \quad r = 1, 2, \ldots, s.$$

This means that $\lambda_j \ (j \neq 0) \geqslant 0$ and $\theta_0^{\sup\ er^*} \delta_i^0 \tilde{\delta}_i^0$ is a feasible solution to (5). But $\theta_0^{\sup\ er^*} \delta_i^0 \tilde{\delta}_i^0 < \theta_0^{\sup\ er^*} \theta_{\mathbf{I}}^{0^*} \leqslant \theta_{\mathbf{I}}^{0^*}$ violating the optimality of $\theta_{\mathbf{I}}^{0^*}$. Thus, if $1 \leqslant \delta_i^0 \tilde{\delta}_i^0 < \theta_{\mathbf{I}}^{0^*}$, then DMU_0 remains extreme-efficient.

Next, if $\delta_i^0 \tilde{\delta}_i^0 = \theta_{\mathbf{I}}^{0^*}$, then we assume DMU_0 is not a frontier point when $\hat{x}_{i0} = \delta_i^0 x_{i0}$ and $\hat{x}_{ij} = x_{ij}/\tilde{\delta}_i^0$, $i \in \mathbf{I}$. Thus, we have $\theta_0^{\sup\ er^*} < 1$ in (3). Now we have $\theta_0^{\sup\ er^*} \delta_i^0 \tilde{\delta}_i^0 \leqslant \theta_0^{\sup\ er^*} \theta_{\mathbf{I}}^{0^*} < \theta_{\mathbf{I}}^{0^*}$ violating the optimality of $\theta_{\mathbf{I}}^{0^*}$. Thus, if $1 \leqslant \delta_i^0 \tilde{\delta}_i^0 \leqslant \theta_{\mathbf{I}}^{0^*}$, then DMU_0 remains on the frontier.

(ii) The proof is similar to (i), but is based on (4) and (6). □

Theorem 3 indicates that the optimal value to a super-efficiency DEA model can actually be decomposed into a data perturbation component (e) for DMU_0 and a data perturbation component ($\tilde{e}$) for the remaining DMUs, $DMU_j (j \neq 0)$. Define

$$\Omega^0 = \begin{cases} \theta_{\mathbf{I}}^{0^*} & \text{if } e = \delta \quad \text{and} \quad \tilde{e} = \tilde{\delta}, \\ \phi_{\mathbf{O}}^{0^*} & \text{if } e = \tau \quad \text{and} \quad \tilde{e} = \tilde{\tau}. \end{cases}$$

Then, the data perturbation can be expressed in a quadratic function,

$$e\tilde{e} = \Omega^0. \tag{7}$$

This function gives an upper bound for input changes and a lower bound for output changes. Fig. 2(a) and (b) illustrates the admissible regions for e and $\tilde{e}$. For example, in Fig. 2(a), since $\delta \geqslant 1$ and $\tilde{\delta} \geqslant 1$, only part of the function $\delta\tilde{\delta} = \theta_{\mathbf{I}}^{0^*}$ forms the upper boundary of a admissible region for δ and $\tilde{\delta}$. Any data variations fall below MN and above lines $\delta = 1$ and $\tilde{\delta} = 1$ will preserve the frontier status of DMU_0. The bigger the $\theta_{\mathbf{I}}^{0^*}$ (or the smaller the $\phi_{\mathbf{O}}^{0^*}$), the larger will the input (output) variation regions be. In fact, the function given by (7) defines the maximum percentage change rates for DMU_0 and $DMU_j \ (j \neq 0)$.

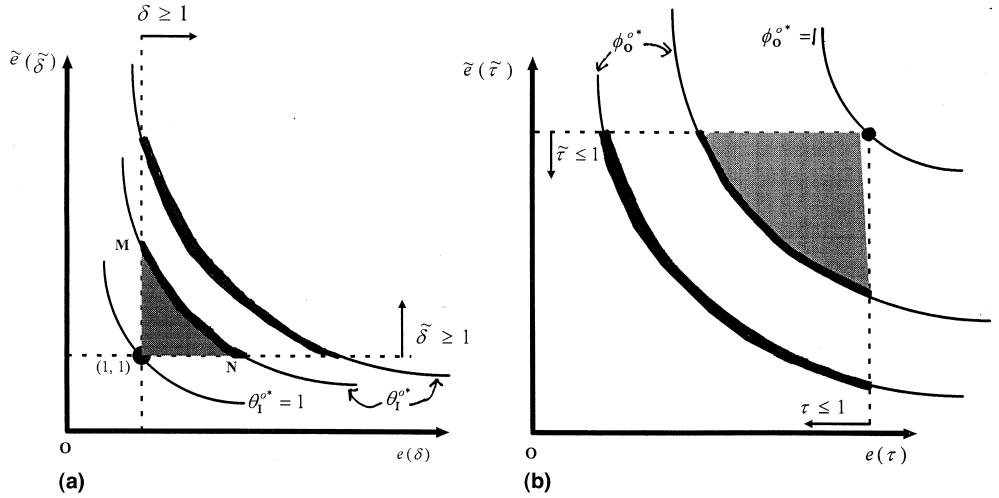

Fig. 2. (a) Input variations; (b) output variations.

Theorem 3 gives sufficient conditions for preserving efficiency. The following theorem implies necessary conditions for preserving efficiency of an extreme-efficient DMU_0.

Theorem 4. *Suppose* $\theta_0^{\text{sup } er^*} > 1$ *and* $\phi_0^{\text{sup } er^*} < 1$, *then*

(i) *If* $\delta_i\tilde{\delta}_i > \theta_{\mathbf{I}}^{0^*}$ *for* $i \in \mathbf{I}$, *then* DMU_0 *will not be extreme-efficient, where* $\theta_{\mathbf{I}}^{0^*}$ *is the optimal value to* (5).

(ii) *If* $\tau_r\tilde{\tau}_r < \phi_{\mathbf{O}}^{0^*}$ *for* $r \in \mathbf{O}$, *then* DMU_0 *will not be extreme-efficient, where* $\phi_{\mathbf{O}}^{0^*}$ *is the optimal value to* (6).

Proof.

(i) We assume that DMU_0 remains extreme-efficient after the data changes in all DMUs with $\delta_i\tilde{\delta}_i > \theta_{\mathbf{I}}^{0^*}$. Consider the input constraints associated with set $\mathbf{I}$ in (5),

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j \frac{x_{ij}}{\tilde{\delta}_i} \leqslant \hat{\theta}_{\mathbf{I}}^{0}\delta_i x_{i0}, \quad i \in \mathbf{I}, \tag{8}$$

where $\hat{\theta}_{\mathbf{I}}^{0}$ is the objective function in (5).

Eq. (8) is equivalent to

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant \hat{\theta}_{\mathbf{I}}^{0}\delta_i\tilde{\delta}_i x_{i0}, \quad i \in \mathbf{I}.$$

Let $\hat{\theta}_{\mathbf{I}}^{0^*}$ be the optimal value. Obviously, $\hat{\theta}_{\mathbf{I}}^{0^*} = \theta_{\mathbf{I}}^{0^*}/\delta_i\tilde{\delta}_i < 1$, where $\theta_{\mathbf{I}}^{0^*}$ is the optimal value to (5). On the basis of Lemma 4 (i), $\hat{\theta}_{\mathbf{I}}^{0^*}$ must be greater than one in (5) with input constraints of (8). A contradiction.

(ii) The proof is similar to (i), but is based on (6). □

Theorem 4 indicates that input (output) data perturbations in all DMUs beyond the variation regions prescribed by function (7) will change the efficiency classification of extreme-efficient DMUs.

Note that $\delta_i\tilde{\delta}_i = \theta_{\mathbf{I}}^{0^*}$ (or $\tau_r\tilde{\tau}_r = \phi_{\mathbf{O}}^{0^*}$) may or may not keep the efficiency classification of an extreme-efficient DMU_0. For example, in Fig. 1, A remains extreme-efficient if $\delta_2\tilde{\delta}_2 = \theta_{\{2\}}^{A^*} = \frac{7}{5}$. (In this situation, A coincides D and both become extreme-efficient.) However, if we consider C and if C's second input is increased to $\theta_{\{2\}}^{C^*}x_{2C} = \frac{3}{2} \times 2 = 3$, then C becomes a member of set F along the ray BB_1 in Fig. 1. (In this situation, we assume $\delta_2 = \theta_{\{2\}}^{C^*} = \frac{3}{2}$ for C and $\tilde{\delta}_2 = 1$ for the remaining DMUs of A, B and D.)

Turning to point A again. If we are only interested in whether a DMU remains on the frontier, rather than in its original efficiency classification, then we may still increase A's second input after A coincides D. We can find this "extra" data perturbation by applying a very small data perturbation to the changed DMU_0 and then applying model (5) or (6). For example, we apply a data perturbation of ε to $\hat{x}_{2A}$ which is the new input value when $\delta_2\tilde{\delta}_2 = \theta_{\{2\}}^{A^*} = \frac{7}{5}$. If we use models (3) and (5), then we know that this changed DMU A with it second input equal to $\hat{x}_{2A} + \varepsilon$ is now in set F and therefore A can still increase it x_2 to any amount larger than 7 and remains on the frontier. Note that, in this case, A may no longer be extreme-efficient. In fact, $\delta_2\tilde{\delta}_2 = \theta_{\{2\}}^{A^*} = \frac{7}{5}$ prescribes a point on line segment AB including A and B. If $\delta_2\tilde{\delta}_2 > \frac{7}{5}$, then A and D switch their positions. Namely, A becomes a weakly efficient DMU and D becomes an extreme-efficient DMU.

Above developments consider the input changes or output changes in all DMUs. Next we consider the following modified DEA measure for simultaneous variations of inputs and outputs

$$\begin{aligned}
\Gamma^* = \min\ & \Gamma \\
\text{s.t.}\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant (1+\Gamma)x_{i0}, \quad i \in \mathbf{I}, \\
& \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, \quad i \notin \mathbf{I}, \\
& \sum_{j=1,\ j\neq 0}^{n} \lambda_j\, y_{rj} \geqslant (1-\Gamma)y_{r0}, \quad r \in \mathbf{O}, \\
& \sum_{j=1,\ j\neq 0}^{n} \lambda_j\, y_{rj} \geqslant y_{r0}, \quad r \notin \mathbf{O}, \\
& \lambda_j\ (j \neq 0) \geqslant 0, \quad \Gamma \text{ unrestricted}.
\end{aligned} \tag{9}$$

If $\mathbf{I} = \{1, 2, \ldots, m\}$ and $\mathbf{O} = \{1, 2, \ldots, s\}$, then (9) is identical to the model of Charnes et al. (1996) when variations in the data are only applied to

DMU_0. Note that if DMU_0 is a frontier point, then $\Gamma \geqslant 0$.

Theorem 5. *Suppose* DMU_0 *is a frontier point. If* $1 \leqslant \delta_i\tilde{\delta}_i \leqslant \sqrt{1+\Gamma^*}$ *and* $\sqrt{1-\Gamma^*} \leqslant \tau_r\tilde{\tau}_r \leqslant 1$, *then* DMU_0 *remains as a frontier point, where* Γ^* *is the optimal value to* (9).

Proof. Equivalently we prove that if $\delta_i\tilde{\delta}_i = \sqrt{1+\Gamma^*}$ and $\tau_r\tilde{\tau}_r = \sqrt{1-\Gamma^*}$, then DMU_0 still remains on the frontier. We shall assume that after the data changes, DMU_0 is a nonfrontier point, and therefore can be enveloped by the adjusted DMU_j $(j \neq 0)$. Thus,

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j \frac{x_{ij}}{\tilde{\delta}_i} \leqslant \delta_i x_{i0}, \quad i \in \mathbf{I},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, \quad i \notin \mathbf{I},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j \frac{y_{rj}}{\tilde{\tau}_r} \geqslant \tau_r y_{r0}, \quad r \in \mathbf{O},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant y_{r0}, \quad r \notin \mathbf{O}.$$

That is,

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant (1+\Gamma^*)x_{i0}, \quad i \in \mathbf{I},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, \quad i \notin \mathbf{I},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant (1-\Gamma^*)y_{r0}, \quad r \in \mathbf{O},$$

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant y_{r0}, \quad r \notin \mathbf{O}.$$

This means that the adjusted DMU_0 with $(1+\Gamma^*)x_{i0}$ $(i \in \mathbf{I})$, x_{i0} $(i \notin \mathbf{I})$, $(1-\Gamma^*)y_{r0}$ $(r \in \mathbf{O})$ and y_{r0} $(r \notin \mathbf{O})$ can be enveloped by the original DMU_j $(j \neq 0)$. However, by Charnes et al. (1996), we know that proportional changes to inputs and outputs, respectively, not beyond the computed values of $(1+\Gamma^*)$ and $(1-\Gamma^*)$ can not change the efficiency of DMU_0 when the remaining DMU_j $(j \neq 0)$ are fixed. Therefore, this leads to a contradiction and completes the proof. □

The result in Theorem 5 generalizes the finding of Charnes et al. (1996) to the situation where variations in the data are applied to all DMUs. Similar to Theorem 4, for an extreme-efficient DMU_0, if $\delta_i\tilde{\delta}_i > \sqrt{1+\Gamma^*}$ and $\tau_r\tilde{\tau}_r < \sqrt{1-\Gamma^*}$, then DMU_0 will not remain extreme-efficient.

3.2. Sensitivity analysis under VRS model

It is obvious that the results in the previous section hold for the VRS frontier DMUs if we add the additional constraint of

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j = 1,$$

respectively, into models (3), (4), (5) and (6).

Because of the translation invariance property resulted from the convex constraint of

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j = 1$$

in the VRS models, we are able to discuss the simultaneous absolute data changes in all DMUs. That is, *absolute data perturbations* (*variations*):

for DMU_0

$$\begin{cases} \hat{x}_{i0} = x_{i0} + \alpha_i, & \alpha_i \geqslant 0,\ i \in \mathbf{I}, \\ \hat{x}_{i0} = x_{i0}, & i \notin \mathbf{I}, \end{cases}$$

and

$$\begin{cases} \hat{y}_{r0} = y_{r0} - \beta_r, & \beta_r \geqslant 0,\ r \in \mathbf{O}, \\ \hat{y}_{r0} = y_{r0}, & r \notin \mathbf{O}; \end{cases}$$

for DMU_j $(j \neq 0)$

$$\begin{cases} \hat{x}_{ij} = x_{ij} - \tilde{\alpha}_i, & \tilde{\alpha}_i \geqslant 0,\ i \in \mathbf{I}, \\ \hat{x}_{ij} = x_{ij}, & i \notin \mathbf{I}, \end{cases}$$

and

$$\begin{cases} \hat{y}_{rj} = y_{rj} + \tilde{\beta}_r, & \tilde{\beta}_r \geqslant 0,\ r \in \mathbf{O}, \\ \hat{y}_{rj} = y_{rj}, & r \notin \mathbf{O}; \end{cases}$$

where $(\hat{})$ represents adjusted data. Note that the data changes defined above are not only applied to all DMUs, but also different in various inputs and outputs. In this case the sensitivity analysis results are also suitable to the additive model.

We modify model (9) to the following linear programming problem:

$$\begin{aligned} \gamma^* = \min\ & \gamma \\ \text{s.t.}\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0} + \gamma, \quad i \in \mathbf{I}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, \quad i \notin \mathbf{I}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant y_{r0} - \gamma, \quad r \in \mathbf{O}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant y_{r0}, \quad r \notin \mathbf{O}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j = 1, \\ & \gamma, \lambda_j\ (j \neq 0) \geqslant 0. \end{aligned} \tag{10}$$

If $\mathbf{I} = \{1, 2, \ldots, m\}$ and $\mathbf{O} = \{1, 2, \ldots, s\}$, then model (10) is used by Charnes et al. (1992) to study the sensitivity of efficiency classifications in the additive model via L_∞ norm when variations in the data are only applied to DMU_0.

Theorem 6. *Suppose* DMU_0 *is a frontier point. If* $0 \leqslant \alpha_i + \tilde{\alpha}_i \leqslant \gamma^*\ (i \in I),\ 0 \leqslant \beta_r + \tilde{\beta}_r \leqslant \gamma^*\ (r \in \mathbf{O})$, *then* DMU_0 *remains as a frontier point, where* γ^* *is the optimal value to* (*10*).

Proof. The proof is similar to that of Theorem 4 by noting that

$$\sum_{j=1,\ j\neq 0}^{n} \lambda_j = 1. \qquad \square$$

If $\mathbf{O} = \emptyset$ then (10) only considers absolute changes in inputs. If $\mathbf{I} = \emptyset$ then (10) only considers absolute changes in output. For different choices of subsets $\mathbf{I}$ and $\mathbf{O}$, we can determine the sensitivity of DMU_0 to the absolute changes of different sets of inputs or (and) outputs where DMU_0's efficiency is deteriorating and DMU_j's $(j \neq 0)$ efficiencies are improving.

We may change the objective function of (10) to "minimize $\sum_{i\in\mathbf{I}} r_i^- + \sum_{r\in\mathbf{O}} \gamma_r^+$" and obtain the following super-efficiency DEA model:

$$\begin{aligned} \min\ & \sum_{i\in\mathbf{I}} \gamma_i^- + \sum_{r\in\mathbf{O}} \gamma_r^+ \\ \text{s.t.}\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0} + \gamma_i^-, \quad i \in \mathbf{I}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j x_{ij} \leqslant x_{i0}, \quad i \notin \mathbf{I}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant y_{r0} - \gamma_r^+, \quad r \in \mathbf{O}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j\ y_{rj} \geqslant y_{r0}, \quad r \notin \mathbf{O}, \\ & \sum_{j=1,\ j\neq 0}^{n} \lambda_j = 1, \\ & \gamma_i^-, \gamma_r^+, \lambda_j\ (j \neq 0) \geqslant 0. \end{aligned} \tag{11}$$

We then obtain a generalized model under L_1 norm. And the results in Charnes et al. (1992) are generalized to the situation of data changes in all DMUs by the following theorem:

Theorem 7. *Suppose* DMU_0 *is a frontier point*. If $0 \leqslant \alpha_i + \tilde{\alpha}_i \leqslant \gamma_i^{-*}\ (i \in \mathbf{I}),\ 0 \leqslant \beta_r + \tilde{\beta}_r \leqslant \gamma_r^{+*}\ (r \in \mathbf{O})$, *then* DMU_0 *remains as a frontier point, where* $\gamma_i^{-*}\ (i \in I)$ *and* $\gamma_r^{+*}\ (r \in \mathbf{O})$ *are optimal values in* (*10*).

Proof. The proof is similar to that of Theorem 4 and is omitted. $\square$

Similar to Theorem 4, for an extreme-efficient DMU_0, if $\alpha_i + \tilde{\alpha}_i > \gamma_i^{-*}$ and $\beta_r + \tilde{\beta}_r > \gamma_r^{+*}$ then DMU_0 will not remain extreme-efficient.

4. Conclusions

The current paper develops a new approach for the sensitivity analysis of DEA models by using various super-efficiency DEA models in which a test DMU is not included in the reference set. The new sensitivity analysis approach simultaneously considers the data perturbations in all DMUs, namely, the change of the test DMU and the changes of the remaining DMUs. The data perturbation in the test DMU and the data perturbation in the remaining DMUs can be different when all remaining DMUs work at improving their efficiencies against the deteriorating of the efficiency of the test efficient DMU. We also develop necessary and sufficient conditions for preserving efficiency when data changes are made for all DMUs. It is obvious that larger optimal values to the super-efficiency DEA models presented in the current study correspond to greater stability of the test DMU in preserving efficiency when the inputs and outputs of all DMUs are changed simultaneously and unequally.

By using super-efficiency DEA models, the sensitivity analysis of DEA efficiency classification can be easily achieved. Since the approach uses optimal values to various super-efficiency DEA models, the results are stable and unique. From the management perspective, our approach provides "what-if" tool to the standard DEA analysis kit. We are able to know what may happen to a DMU's efficiency, if data variation occurs in all DMUs as a result of new strategic planning.

By the additional constraint on $\sum_{j=1,\ j\neq 0}^{n} \lambda_j$, the approach can easily be modified to study the sensitivity of other DEA models that satisfy different returns to scale (Seiford and Thrall, 1990; Charnes et al., 1994; Färe et al., 1994). The new sensitivity analysis technique can well be applied to inefficient DMUs if we are interested in preserving the inefficiency of inefficient DMUs.

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