Short Communication
Integrated data envelopment analysis: Global vs. local optimum
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Abstract

Chiou et al. (2010) (a joint measurement of efficiency and effectiveness for non-storable commodities: integrated data envelopment analysis approaches. European Journal of Operational Research 201, 477–489) propose an integrated data envelopment analysis model in measuring decision making units (DMUs) having a two-stage internal network structure with multiple inputs, outputs, and consumptions. They claim that any optimal solution determined by their DEA model is a two-stage process, each stage having the DEA ratio efficiency based upon the fractional linear model of Charnes et al. (1978) or Banker et al. (1984). The modeling difference between Chiou et al. (2010) and Kao and Hwang (2008) is that the former uses the arithmetic mean of two efficiency ratios and the latter uses the geometric mean. Unlike the use of arithmetic mean, the use of geometric mean of the two efficiency ratios allows us to convert the DEA model into a linear program. Although the use of arithmetic mean of efficiency ratios cannot be converted into a linear DEA model, Liang et al. (2006) show that such a non-linear DEA model can be converted into a parametric linear program whose global optimal solution can be approximated by solving a sequence of linear programs combined with a simple search algorithm. However, Chiou et al. (2010) do not adopt the approach of Liang et al. (2006) and claim that any local optimum from their model is the global optimum. We show that Chiu et al.’s (2010) claim is erroneous, because they do not use the Hessian matrix correctly in examining the concavity of their DEA model’s objective function. We show that their DEA model’s objective function (to be maximized) is not concave. As a result, their DEA model is unsuitable in practice due to a lack of efficient algorithm for this particular non-convex DEA model. We further show that Chiou et al.’s (2010) model is a special case of a well-known two-stage network DEA model, and it can be transformed into a parametric linear program for which an approximate global optimal solution can be obtained by solving a sequence of linear programs in combination with a simple search algorithm.

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1. Introduction

Using an example of measuring relative performance of non-storable transport services, Chiou et al. (2010) develop an integrated data envelopment analysis (DEA) for situations where decision making units (DMUs) have a two-stage process. In their case, the first stage uses inputs (e.g., labor and vehicles) to produce outputs (e.g., vehicle miles), and the second stage then uses the first stage outputs to generate consumptions (e.g., passenger-miles and revenue). Similar two-stage processes have been studied in other applications, e.g., insurance companies by Kao and Hwang (2008). In such a two-stage process, each stage has its own DEA ratio efficiency based upon the fractional linear model of Charnes et al. (1978) or Banker et al. (1984). The modeling difference between Chiou et al. (2010) and Kao and Hwang (2008) is that the former uses the arithmetic mean of the two efficiency ratios and the latter uses the geometric mean. Unlike the use of arithmetic mean, the use of geometric mean of the two efficiency ratios allows us to convert the DEA model into a linear program.

Although the use of arithmetic mean of efficiency ratios cannot be converted into a linear DEA model, Liang et al. (2006) show that such a non-linear DEA model can be converted into a parametric linear program whose global optimal solution can be approximated by solving a sequence of linear programs combined with a simple search algorithm. However, Chiou et al. (2010) do not adopt...
2. Model

Following the notations from Chiou et al. (2010), assume that there are \( I \) DMUs and each DMU uses \( J \) inputs to produce \( R \) outputs which are transformed into \( S \) consumptions. It is assumed that \( J \geq 1, R \geq 1, \) and \( S \geq 1 \). Using a vector notation for inputs, outputs, and consumptions, we say that DMU \( k \) \( (k = 1, 2, \ldots, I) \) uses a vector of inputs \( x_k = \left( x_{k1}, \ldots, x_{kr} \right)^T \in \mathbb{R}^r \) to produce a vector of outputs \( y_k = \left( y_{kl}, \ldots, y_{kr} \right)^T \in \mathbb{R}^r \) that are transformed into a vector of consumptions \( z_k = \left( z_{kl}, \ldots, z_{kr} \right)^T \in \mathbb{R}^r \). The integrated CRS DEA model proposed by Chiou et al. (2010) is given by

\[
\begin{align*}
\max_{w, v, u} & \quad E_k = \left( \sum_{j=1}^{R} u_j y_{kj} \right) + \left( \sum_{i=1}^{S} w_i z_{ki} \right) \\
\text{s.t.} & \quad \sum_{r=1}^{J} u_j x_{ij} \leq v_i, \quad i = 1, 2, \ldots, I, \\
& \quad \sum_{s=1}^{S} w_i z_{is} \leq u_r, \quad i = 1, 2, \ldots, I, \\
& \quad v_j \geq 0, \quad j = 1, 2, \ldots, J, \\
& \quad w_s \geq 0, \quad s = 1, 2, \ldots, S, \\
& \quad u_r \geq 0, \quad r = 1, 2, \ldots, R,
\end{align*}
\]

where \( u_j, u_r, \) and \( w_s \) are the weights given to input \( j \), output \( r \), and consumption \( s \), respectively, chosen by DMU \( k \).

We should point out that Chiou et al. (2010) describe a situation where DMUs have a two-stage process depicted in Fig. 1. In the DEA literature, the outputs from the first stage are usually called intermediate measures and the consumptions from the second stage are called outputs. Since linear-fractional programming belongs to the class of quasi-convex optimization problems, it suffices to check the concavity of the objective function to ensure convex optimization. The second-order condition for convexity of a twice differentiable function is that the Hessian (i.e., the square matrix of second-order partial derivatives) of the function is negative semi-definite.

Chiou et al. (2010) write they examine the bordered Hessian (i.e., the Hessian bordered with first-order partial derivatives of the constraints) to claim the “uniqueness”\(^3\) of model (1) in Section 3.2.1 of their paper. Their discussion therein, however, is totally irrelevant. As is well known, a bordered Hessian is used for the second-derivative test in certain constrained optimization problems, which is based on the Lagrange multiplier method. It tests whether a given feasible solution is locally optimal or not for the given constrained optimization problem. It has nothing to do with the purpose of ensuring convex optimization. Furthermore, we should point out that although Chiou et al. (2010, Section 3.2.1, p. 481) claim that their Hessian is a bordered Hessian, it is actually just the Hessian of the objective function.

The Hessian of the objective function, with the order of variables being \((w, v, u)\) is

\[
H(E_k) = \begin{pmatrix}
0 & 0 & -x_k^T \\
0 & 2u^T y_k & x_k x_k^T \\
-x_k^T & x_k x_k^T & 2w^T z_k y_k y_k^T
\end{pmatrix}
\]

where \( u = (u_1, \ldots, u_J)^T, v = (v_1, \ldots, v_S)^T, \) and \( w = (w_1, \ldots, w_R)^T \). Zeros in the matrix are of appropriate dimension. To claim the negative semi-definiteness of the Hessian, Chiou et al. (2010) examine all leading principal minors. This is, however, a false conclusion. To test for negative semi-definiteness of a matrix, it is not sufficient to examine all leading principal minors; instead, all principal minors should be examined.\(^4\) More specifically, for a real \( n \times n \) square matrix to be negative semi-definite, principal minors of order \( k \) must be non-negative if \( k \) is an odd number and nonnegative if \( k \) is an even number, for any \( k \leq n \). The first nonzero \( J \times J \) block matrix on the diagonal of \( H \) involves \( x_k x_k^T \), which is a rank one matrix assuming \( x_k \neq 0 \), and at least one of its diagonal elements is positive. In other words, there exists at least one \( 1 \times 1 \) principal minor in \( H \) which is positive. This excludes the possibility of negative semi-definiteness of \( H \), and it can be concluded that the objective function is not concave. Therefore, model (1) is not a convex optimization, implying a lack of efficient algorithms for determining a globally optimal solution.

4. Determining a global optimal solution

As seen in the above, model (1) is non-convex optimization. Non-convex optimization problems are hard to solve to optimality.
in general, and there may exist local optima which are not global optimal. It is theoretically and practically difficult to check whether a given local optimal solution is globally optimal, and this prevents the development of efficient solution methods.

However, note that Chiou et al.'s (2010) model is actually a special case of Liang et al. (2006) if we remove additional inputs to stage 2 (see Fig. 1). Liang et al. (2006) develop a solution method to obtain an approximate global optimal solution when DEA model's objective function is the sum of two efficiency ratios as in model (1).

First, we rewrite model (1) in vector form:
\[
\begin{align*}
\max_{u,v,w} & \quad E_k = \frac{u^T y_k}{u^T x_k} + \frac{w^T z_k}{w^T y_k} \\
\text{s.t.} & \quad u^T y_i \leq v^T x_i, \quad i = 1, 2, \ldots, I, \\
& \quad w^T z_i \leq u^T y_i, \quad i = 1, 2, \ldots, I, \\
& \quad v \geq 0, \quad w \geq 0, \quad u \geq 0.
\end{align*}
\] (2)

Next we apply the following Charnes–Cooper transformation to model (2):
\[
\begin{align*}
t_1 &= \frac{1}{w^T x_k}, \quad t_2 = \frac{1}{u^T y_k}, \\
\omega &= t_1 v, \quad c_1 = t_1 u, \\
\mu &= t_2 w, \quad c_2 = t_2 u.
\end{align*}
\]

Note that in the above transformation, \(c_1 = t_1 u\) and \(c_2 = t_2 u\) imply a linear relationship between \(c_1\) and \(c_2\). Therefore, we can assume \(c_2 = hc_1, h > 0\), where \(h\) is a positive real number. Then model (2) can be transformed into:
\[
\begin{align*}
\max_{\mu,c_1,c_2,h} & \quad E_k = c_1^T y_k + \mu^T z_k \\
\text{s.t.} & \quad c_1^T y_i \leq \omega^T x_i, \quad i = 1, 2, \ldots, I, \\
& \quad \mu^T z_i \leq c_2^T y_i, \quad i = 1, 2, \ldots, I, \\
& \quad \omega^T x_k = 1, \\
& \quad c_1^T y_k = 1, \\
& \quad c_2 = hc_1, \\
& \quad \mu \geq 0, \quad \omega \geq 0, \quad c_1 \geq 0, \quad c_2 \geq 0, \quad h > 0,
\end{align*}
\] (3)

which can be further reduced to the following by eliminating \(c_2\):
\[
\begin{align*}
\max_{\mu,c_1,h} & \quad E_k = c_1^T y_k + \mu^T z_k \\
\text{s.t.} & \quad c_1^T y_i \leq \omega^T x_i, \quad i = 1, 2, \ldots, I, \\
& \quad \mu^T z_i \leq hc_1^T y_i, \quad i = 1, 2, \ldots, I, \\
& \quad \omega^T x_k = 1, \\
& \quad hc_1^T y_k = 1, \\
& \quad \mu \geq 0, \quad \omega \geq 0, \quad c_1 \geq 0, \quad h > 0.
\end{align*}
\] (4)

Taking \(h\) as a parameter, model (4) can be considered a parametric linear program, and it can be solved to any desired tolerance via a simple line search method over a certain range of \(h\).

Liang et al. (2006) suggest a lower bound and an upper bound on \(h\) for the line search. Their upper bound is the output-oriented efficiency score from their leader–follower model when stage 2 in Fig. 1 (“outputs” as DEA inputs and “consumptions” as DEA outputs) is the leader. Their lower bound is obtained using the fact \(c_1^T y_k \leq \omega^T x_k\) and \(c_1^T x_k = 1\). That is, \(h \geq 1\).

We can further tighten their lower bound due to the simplified two-stage process in Fig. 1, by which the search space for the optimal \(h\) can be narrowed down. We notice that \(h = 1/c_1^T y_k\), and it indicates that \(h\) is greater than or equal to the standard output-oriented CRS efficiency score of the first stage (involving only “inputs” as DEA inputs and “outputs” as DEA outputs). This is because the input-oriented CRS model constrains the weighted inputs to be one for a DMU under evaluation, and the output-oriented CRS score is the inverse of the input-oriented CRS score.

5. Concluding remarks

We have shown that Chiou et al.’s (2010) model is a non-convex optimization problem, and their claim that any optimal solutions from model (1) are globally optimal is false. We have also developed a procedure to obtain an approximate global optimal solution to model (1) based on Liang et al. (2006) with improved search bounds.

Given the above situation and given the fact that Chiou et al.’s (2010) approach is based upon a two-stage network DEA structure, in addition to our proposed solution method under CRS, we recommend using other existing DEA approaches for evaluating two-stage structures. For example, under CRS (or Charnes et al. (1978)), we can use the approaches developed by Kao and Hwang (2008) and Liang et al. (2008) where geometric mean of the two ratio efficiencies is used. Under the situation of VRS (or Banker et al. (1984)), we can use the approach developed by Chen et al. (2009) where weighted average of the ratio efficiencies is used.

Finally, although our discussion is based upon the assumption of CRS, the same conclusion can be obtained for the case of VRS. In other words, Chiou et al.’s (2010) DEA model under VRS is also a non-convex problem. As a result, their discussions in the application (Section 4) and case study (Section 5) are invalid.

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