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Decision Support

Super-efficiency DEA in the presence of infeasibility

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ABSTRACT

It is well known that super-efficiency data envelopment analysis (DEA) approach can be infeasible under the condition of variable returns to scale (VRS). By extending of the work of Chen (2005), the current study develops a two-stage process for calculating super-efficiency scores regardless whether the standard VRS super-efficiency mode is feasible or not. The proposed approach examines whether the standard VRS super-efficiency DEA model is infeasible. When the model is feasible, our approach yields super-efficiency scores that are identical to those arising from the original model. For efficient DMUs that are infeasible under the super-efficiency model, our approach yields super-efficiency scores that characterize input savings and/or output surpluses. The current study also shows that infeasibility may imply that an efficient DMU does not exhibit super-efficiency in inputs or outputs. When infeasibility occurs, it can be necessary that (i) both inputs and outputs be decreased to reach the frontier formed by the remaining DMUs under the input-orientation and (ii) both inputs and outputs be increased to reach the frontier formed by the remaining DMUs under the output-orientation. The newly developed approach is illustrated with numerical examples.

1. Introduction

Data envelopment analysis (DEA) measures the relative efficiencies of peer decision making units (DMUs) that have multiple input and outputs. DMUs that receive a score of unity are deemed as on the DEA (best-practice) frontier. To break the tie of efficient DMUs, the CCR model of Charnes et al. (1978) is modified by Andersen and Petersen (1993). This modified CCR model is called super-efficiency model where a DMU under evaluation is excluded from the reference set. For inefficient DMUs, the super-efficiency model yields the identical standard DEA score. However, for efficient DMUs, super-efficiency scores are not less than one under the assumption of input-orientation, for example.

The CCR model is under the condition of constant returns to scale (CRS). While the super-efficiency model under CRS does not suffer the problem of infeasibility, the super-efficiency model under the condition of variable returns to scale (VRS) can be infeasible. Seiford and Zhu (1999) provide the necessary and sufficient conditions for infeasibility of super-efficiency models, and further show that infeasibility must occur in the case of the variable returns to scale (VRS) super-efficiency model.

A number of studies have tried to solve the problem of VRS super-efficiency model's infeasibility. Lovell and Rouse (2003) suggest using a user-defined scaling factor to make the VRS super-efficiency model feasible. Yet, as indicated in Cook et al. (2009), it is possible that Lovell and Rouse's (2003) approach assigns the user-defined scaling factor as the super-efficiency score for all DMUs having infeasible solutions. Cook et al. (2009) develop a modified VRS super-efficiency model for efficient DMUs that are infeasible under the standard VRS super-efficiency model. Cook et al. (2009) further define a super-efficiency score with respect to both input and output super-efficiencies.

In fact, as pointed out by Chen (2005), one needs to use both input- and output-oriented super-efficiency models to fully characterize the super-efficiency when infeasibility occurs. Chen (2005) further suggests that one should integrate the input and output super-efficiency scores by solving both the input- and output-oriented VRS super-efficiency models.

The current study extends the work of Chen (2005) by proposing a two-stage super-efficiency calculation. We find that the infeasibility of input-oriented super-efficiency occurs when the outputs of the evaluated DMU is outside the production possibility set spanned by the outputs of the remaining DMUs and the infeasibility of output-oriented super-efficiency occurs when the inputs of the evaluated DMU is outside the production possibility set spanned by the inputs of the remaining DMUs. As indicated in Seiford and Zhu (1999) and Chen (2005), infeasibility in the input-oriented super-efficiency can indicate that a particular efficient DMU under evaluation exhibits super-efficiency performance only in outputs. Infeasibility in the output-oriented super-efficiency can indicate that a particular efficient DMU under evaluation exhibits super-efficiency performance only in inputs. Chen (2005) points out that super-efficiency can be regarded as input saving/output surplus

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achieved by an efficient DMU. Therefore, in the first stage, the current study seeks to simultaneously test whether a VRS super-efficiency model is infeasible, and detect output surpluses (input savings) when infeasibility occurs in the input-oriented (output-oriented) VRS super-efficiency model. Then, in a second stage calculation, a modified VRS super-efficiency model is proposed to calculate the super-efficiency for all the efficient DMUs.

If super-efficiency only exists in inputs (or outputs), then our modified output-oriented (or input-oriented) super-efficiency model may actually indicates inefficient performance. In other words, infeasibility may imply inefficient performance. This is consistent with the findings in Chen (2005) and Cook et al. (2009).

Like the approach in Cook et al. (2009), when infeasibility occurs, our approach may require that (i) both inputs and outputs be decreased to reach the frontier formed by the remaining DMUs under the input-orientation and (ii) both inputs and outputs be increased to reach the frontier formed by the remaining DMUs under the output-orientation.

The proposed new model provides VRS super-efficiency scores that are equivalent to those arising from the VRS super-efficiency model when feasibility is present. When the VRS super-efficiency model is infeasible, our new model determines a referent (benchmark) DMU formed by the remaining DMUs and yields a score that characterizes the super-efficiency in inputs and outputs. We also show that the referent DMU is on the frontier formed by the remaining DMUs. The current paper proposes ways to fully integrate input and output super-efficiencies when infeasibility presents. This extends the results of Cook et al. (2009).

The rest of the paper is organized as follows. Section 2 presents preliminaries for developing the new approach. Section 3 develops our super-efficiency DEA approach in the presence of infeasibility. Section 4 applies the newly developed approach to data on the 20 largest Japanese companies and 15 US cities that are used in Chen (2004) and Cook et al. (2009). We further demonstrate how our new proposed approach works and what infeasibility implies. Conclusions are presented in Section 5.

2. Preliminaries

Suppose we have a set of n DMUs, $\{DMU_j : j = 1,2,\ldots,n\}$. Let (x^k,y^k) denote the input and output vectors of the kth DMU. The ith input of the kth DMU is denoted as x_i^k and the rth output of the kth DMU is denoted as y_r^k.

Arranging the data sets in matrices $X = (x^j)$ and $Y = (y^j)$ $(j = 1,\ldots,n)$, the production possibility set spanned by (X,Y) with VRS can be written as $PPS(X,Y) = \{(x,y)|X\lambda \leqslant x, Y\lambda \geqslant y, e\lambda = 1, \lambda \geqslant 0\}$, where e denotes a row vector in which all elements are equal to 1. Without loss of generality, we shall denote a production possibility set by the capital P throughout the paper. As indicated in the conventional definition of the production possibility set, $(x,y) \in P$ means that x can produce y.

Definition 1. Production possibility set of input spanned by X with VRS is $PPS(X) = \{x|X\lambda \leqslant x, e\lambda = 1, \lambda \geqslant 0\}$.

Definition 2. Production possibility set of output spanned by Y with VRS is $PPS(Y) = \{y|Y\lambda \geqslant y, e\lambda = 1, \lambda \geqslant 0\}$.

Definition 3. The input production set that dominates x^k is denoted as $DN(x^k) = \{x|x \leqslant x^k\}$.

Definition 4. The output production set that dominates y^k is denoted as
$DN(y^k) = \{y|y \geqslant y^k\}$.

Definition 5 (*Domination*). A point $p = (x,y)$ dominates $q = (x',y')$ if $x \leqslant x'$ and $y \geqslant y'$. A point $p = (x,y)$ strictly dominates $q = (x',y')$ if $x < x'$ and $y > y'$. A point $p = (x,y)$ semi-strictly dominates $q = (x',y')$ if $p = (x,y)$ dominates $q = (x',y')$ and $p \neq q$.

Definition 6 (*Non-domination*). A point $p = (x,y)$ is a non-dominated point in P if there is no point $p' = (x',y')$ in P such that $p \neq p'$ and p' dominates p.

Definition 7 (*Pareto y-highest*). A point $p = (x,y)$ is said to be Pareto y-highest in P if there is no point $p' = (x',y')$ in P other than $p = (x,y)$ such that $y \leqslant y'$ and $y \neq y'$.

That a point $p = (x,y)$ in P is said to be Pareto y-highest implies that some of the y-coordinates of $p = (x,y)$ are largest in comparison with the points in P.

Definition 8 (*Pareto x-lowest*). A point $p = (x,y)$ is said to be Pareto x-lowest in P if there is no point $p' = (x',y')$ in P other than $p = (x,y)$ such that $x \geqslant x'$ and $x \neq x'$.

That a point $p = (x,y)$ in P is said to be Pareto x-lowest implies that some of the x-coordinates of $p = (x,y)$ are lowest in comparison with the other points in P.

3. Super-efficiency DEA in the presence of infeasibility

The input-oriented VRS super-efficiency model for efficient DMU_k can be expressed as

$$
\begin{aligned}
\min\ & \theta \\
\text{s.t.}\ & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j \leqslant \theta x_i^k \quad i = 1,2,\ldots,m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j \geqslant y_r^k \quad r = 1,2,\ldots,s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 \\
& \lambda_j \geqslant 0, \quad j \neq k
\end{aligned}
\tag{1}
$$

It is obvious that (1) is infeasible when $y^k \notin PPS(Y)$ where $Y = (y^j)$ $(j = 1,\ldots,n, j \neq k)$. Chen (2005) points out that if model (1) is feasible, then the optimal θ^* represents the input saving of DMU_k compared to the frontier formed by the remaining DMUs. Seiford and Zhu (1999) and Chen (2005) further point out that infeasibility of model (1) may be due to the fact that the DMU under evaluation does not exhibit input saving and only exhibit output super-efficiency, or output surplus. We therefore consider the following linear programming problem which seeks to determine potential surpluses in individual outputs.

$$
\begin{aligned}
\min\ & \sum_{r=1}^{s} s_r \\
\text{s.t.}\ & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j + s_r y_r^k \geqslant y_r^k \quad r = 1,2,\ldots,s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 \\
& \lambda_j \geqslant 0, \quad j \neq k \\
& s_r \geqslant 0, \quad r = 1,2,\ldots,s
\end{aligned}
\tag{2}
$$

We note that model (2) is similar to the Seiford and Zhu's (1999) model for testing infeasibility. Our model (2) can be viewed as a slack-based and units invariant version of Seiford and Zhu's (1999) radial model. The SBM proposed by Tone (2001) is similar to Russell measure (1985,1988) in that both deal with slacks and give an efficiency measure between 0 and 1. Fukuyama and Weber (2009) proposed a directional slacks-based measure of technical inefficiency with the intention to generalize some of the existing slacks-based measures of inefficiency. Based on the notion of the range of possible improvement, Portela et al. (2004) proposed the range directional model which is translation invariant and units invariant.

Theorem 1. *Let $(s_1^*,\ldots,s_s^*)$ denote a set of optimal solution in (2). Then model (1) is feasible if and only if $s_r^*=0$ for $r = 1,\ldots,s$.*

Proof. Let $(\lambda_1^*,\ldots,\lambda_n^*,s_1^*,\ldots,s_s^*)$ be a set of optimal solution in model (2). If $s_r^*=0$ for $r = 1,\ldots,s$, there exists θ' such that

$$\begin{aligned}
&\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j^* x_i^j \leqslant \theta' x_i^k \quad i=1,2,\ldots,m\\
&\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j^* y_r^j \geqslant y_r^k \quad r=1,2,\ldots,s\\
&\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j^* = 1\\
&\theta' \geqslant 0\\
&\lambda_j^* \geqslant 0,\quad j\neq k
\end{aligned}$$

Hence model (1) has feasible solution. If model (1) is feasible, we have one solution $(\theta',\lambda_1',\ldots,\lambda_n')$ such that

$$\begin{aligned}
&\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j' x_i^j \leqslant \theta' x_i^k \quad i=1,2,\ldots,m\\
&\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j' y_r^j \geqslant y_r^k \quad r=1,2,\ldots,s\\
&\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j' = 1\\
&\theta' \geqslant 0\\
&\lambda_j' \geqslant 0,\quad j\neq k
\end{aligned}$$

Therefore, $s_r^*=0$ for $r = 1,\ldots,s$ are in the optimal solution of (2). □

Theorem 1 indicates that the input-oriented VRS super-efficiency model is infeasible if and only if there exists some $s_r^*>0$. Note that these $s_r^* y_r^k$ are not the output slacks in the standard DEA approach, but represent the output surpluses in DMU_k compared to the frontier formed by the rest of DMUs.

We then establish the following modified VRS super-efficiency model which is unit invariant.

$$\begin{aligned}
\min\ & \hat{\theta}\\
\text{s.t.}\ & \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j x_i^j \leqslant \hat{\theta} x_i^k \quad i=1,2,\ldots,m\\
& \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j y_r^j + s_r^* y_r^k \geqslant y_r^k \quad r=1,2,\ldots,s\\
& \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j = 1\\
& \lambda_j \geqslant 0,\quad j\neq k
\end{aligned}\tag{3}$$

where $(s_1^*,\ldots,s_s^*)$ are optimal solutions in model (2).

Let $\hat{\theta}^*$ be the optimal solution of (3) and θ^* be the optimal solution of (1). If model (1) is feasible, then obviously $\hat{\theta}^*=\theta^*$, indicating that model (3) yields the identical super-efficiency score when model (1) is feasible.

We next show that projection or benchmark for DMU_k is always on the frontier of the remaining DMUs, and unlike model (1), model (3) is always feasible.

Theorem 2. *The DEA projection or benchmark based upon model (3) is either (i) a Pareto x-lowest non-dominated point in $\{(x,y)|(x,y)\in PPS(X,Y)$ and $y \geqslant y^k\}$ if $y^k \in PPS(Y)$ or (ii) a Pareto y-highest non-dominated point in PPS(X,Y) if $y^k \notin PPS(Y)$, where $X = (x^j)$ $(j = 1,\ldots,n, j \neq k)$ and $Y = (y^j)$ $(j = 1,\ldots,n, j \neq k)$.*

Proof. (i) Since $y^k \in PPS(Y)$, model (1) is feasible. Based upon Theorem 1, if (1) is feasible, then $s_r^*=0$ for $r = 1,\ldots,s$ in (2). Hence (3) is equivalent to (1).

The projection $(\hat{x}^k,\hat{y}^k)$ of (x^k,y^k) can be obtained via model (1) as $\hat{x}_i^k=\sum_{j=1, j\neq k}^{n}\lambda_j^* x_i^j$ and $\hat{y}_r^k=\sum_{j=1, j\neq k}^{n}\lambda_j^* y_r^j$. It is obvious that $\hat{y}^k \geqslant y^k$ and $(\hat{x}^k,\hat{y}^k)\in PPS(X,Y)$. Let $P = \{(x,y)|(x,y) \in PPS(X,Y)$ and $y \geqslant y^k\}$. Then $(\hat{x}^k,\hat{y}^k)\in P$.

If $(\hat{x}^k,\hat{y}^k)$ is not a non-dominated point in P, there exists a point $(x',y') \in P$ such that (x',y') dominates $(\hat{x}^k,\hat{y}^k)$, indicating that θ^* is not the best solution. Therefore, $(\hat{x}^k,\hat{y}^k)$ is a non-dominated point in P.

Moreover, $(\hat{x}^k,\hat{y}^k)$ is a x-lowest point in P. If $(\hat{x}^k,\hat{y}^k)$ is not a x-lowest point in P, there exists a point $(x',y') \in P$ such that $x' \leqslant \hat{x}^k$ and $x' \neq \hat{x}^k$, indicating that (x',y') is a non-dominated point in P and θ^* is not the optimal solution. Therefore, $(\hat{x}^k,\hat{y}^k)$ is a x-lowest point in P.

(ii). Let $\hat{\theta}^*$ be the optimal solution of (3) and s_r^* be the optimal solutions of (2). Since $y^k \notin PPS(Y)$, there exist some $s_r^*>0$. The projection of (x^k,y^k) can be identified by solving the following model:

$$\begin{aligned}
\max\ & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\\
\text{s.t.}\ & \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j x_i^j + s_i^- = \hat{\theta}^* x_i^k \quad i=1,2,\ldots,m\\
& \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j y_r^j - s_r^+ + s_r^* y_r^k = y_r^k \quad r=1,2,\ldots,s\\
& \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j = 1\\
& s_i^- \geqslant 0 \quad i=1,2,\ldots,m\\
& s_r^+ \geqslant 0 \quad r=1,2,\ldots,s\\
& \lambda_j \geqslant 0,\quad j\neq k
\end{aligned}$$

Then the projection of (x^k,y^k) is $(\hat{x}^k,\hat{y}^k)$ where $\hat{x}_i^k=\sum_{j=1, j\neq k}^{n}\lambda_j^* x_i^j$ and $\hat{y}_r^k=\sum_{j=1, j\neq k}^{n}\lambda_j^* y_r^j$. It is obvious that $(\hat{x}^k,\hat{y}^k)\in PPS(X,Y)$. If $(\hat{x}^k,\hat{y}^k)$ is not a non-dominated point in $PPS(X,Y)$, there exists a point $(x',y') \in PPS(X,Y)$ such that (x',y') dominates $(\hat{x}^k,\hat{y}^k)$, indicating that either s_r^* for $r = 1,\ldots,s$ or $\hat{\theta}^*$ is not the optimal solution. Therefore, $(\hat{x}^k,\hat{y}^k)$ is a non-dominated point in $PPS(X,Y)$. Since $s_{r'}^*>0$, we have $s_{r'}^{+^*}=0$. Assume there is a point $(x',y') \in PPS(X,Y)$ such that $y'_{r'}>\hat{y}_{r'}^k$, indicating that $s_{r'}^{+^*}>0$, a contradiction. Therefore, (x',y') does not exist. and $(\hat{x}^k,\hat{y}^k)$ is a y-highest point in $PPS(X,Y)$. □

Theorem 3. *Model (3) is always feasible.*

Proof. Let $X = (x^j)$ $(j = 1,\ldots,n, j \neq k)$ and $Y = (y^j)$ $(j = 1,\ldots,n, j \neq k)$.

(A) Assume that $(x^k, y^k) \in PPS(X, Y)$, which implies that

$$(x^k, y^k) \in \left\{ (x, y) | x_i \geqslant \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j,\ y_r \leqslant \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j, \right.$$
$$x = (x_1, \ldots, x_i, \ldots, x_m),\ y = (y_1, \ldots, y_r, \ldots y_s),$$
$$\left. \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0 \right\}$$

In other words, $x_i^k \geqslant \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j$ and $y_r^k \leqslant \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j$. Hence the model (2) is feasible and the optimal solution for (2) is $s_r^* = 0$ for $r = 1,\ldots,s$. Model (3) is also feasible.

(B) Assume that $(x^k, y^k) \notin PPS(X, Y)$, which implies that

$$(x^k, y^k) \notin \left\{ (x, y) | x_i \geqslant \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j,\ y_r \leqslant \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j, \right.$$
$$x = (x_1, \ldots, x_i, \ldots, x_m),\ y = (y_1, \ldots, y_r, \ldots y_s),$$
$$\left. \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0 \right\}$$

So either $\exists i : \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j > x_i^k$ or $\exists r : \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j < y_r^k$. Hence (2) and (3) are feasible if $\theta > 1$ when $\exists i : \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j > x_i^k$ or $s_r > 0$ when $\exists r : \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j < y_r^k$. □

One would expect that for efficient DMUs, their input-oriented super-efficiency scores should be greater than one. Such expectation is realistic for the CRS assumption. Under VRS assumption, because of the possible infeasibility, such expectation may not be met due to the fact that an efficient DMU needs to decrease both its inputs and outputs to reach the frontier formed by the rest of DMUs. To further illustrate this point, we consider a simple numerical example shown in Fig. 1 where we have three efficient DMUs, A(1,1), B(2,3) and C(4,4).

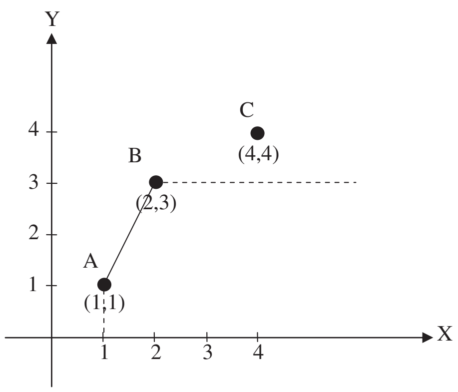

Fig. 1. A simple example.

For DMU C we have infeasibility in model (1), $s^* = 1/4$ in model (2) and $\hat{\theta}^* = 0.5 (< 1)$ in model (3). This is because DMU B is identified as its benchmark. To reach DMU B, it has to decrease its both input and output. It is reasonable to have DMU B's super-efficiency score greater than 1 for two reasons. The first is that DMU B is efficient in the original BCC model. The second is that DMU B is outside the efficient boundary formed by remaining DMUs as shown in Fig. 1. Note that DMU B is above the horizontal dashed line. If DMU B is moving downward onto the dashed line, what should the super-efficiency score of DMU B be? In the original BCC model, a DMU should have its radial score less than 1 indicating it is inefficient. Therefore, its supper-efficiency should be less than 1 if it is on the dashed line. Based on the rationale above, we want to devise a composite score so that the score of a DMU is greater than 1 if it is above the dashed line and its score will decrease when it is moving downward or rightward.

To address such an issue, we modify the super-efficiency score obtained from model (3) in the following manner.

$$\breve{\theta} = \begin{cases} \dfrac{\sum_{r\in R}\left(\frac{y_r^k}{y_r^k - s_r^* y_r^k}\right)}{|R|} + \hat{\theta}^*, & \text{if } R \neq \phi \\ \hat{\theta}^*, & \text{if } R = \phi \end{cases}$$

where $R = \{r | s_r^* > 0\}$ based upon model (2) and |R| is the cardinality of the set R.

The efficiency measure consists of two parts, in which $\frac{\sum_{r\in R}\left(\frac{y_r^k}{y_r^k - s_r^*}\right)}{|R|}$ reflects how far the DMU k is above the dashed efficient boundary and $\hat{\theta}^*$ reflects the input excess if it is less than 1 or input savings if it is greater than 1. When a DMU falls in the area north of the dashed line in Fig. 1, like unit C, its efficiency of the original BCC is 1 which implies that its super efficiency should be greater than 1, which is guaranteed in our measure by $\frac{\sum_{r\in R}\left(\frac{y_r^k}{y_r^k - s_r^*}\right)}{|R|}$. If a DMU is above the dashed line and we move it to the right horizontally, its efficiency should decrease. This is reflected by $\hat{\theta}^*$

For DMU C, we now have $\breve{\theta} = \frac{11}{6} > 1$, as its modified input-oriented super-efficiency score. For DMU C, when the input-oriented super-efficiency model (1) is infeasible, from Fig. 1 it is clear that DMU C does not have input super-efficiency. DMU C only has super-efficiency in its output. Our above proposed modification integrates both input and output super-efficiency when infeasibility occurs. For example, in the original BCC model, the efficiency of DMU C is 1. The super efficiency of C should be at least 1. Because it is above horizontal dashed line, the super efficiency of C should be greater than 1. Since $R = \{1\}$ and $s_1^* = 1$, $\frac{\sum_{r\in R}\left(\frac{y_r^k}{y_r^k - s_r^*}\right)}{|R|} = \frac{\left(\frac{4}{4-1}\right)}{1} = \frac{4}{3}$ which accounts for this. Since DMU C is to the right of its benchmark B which is the same benchmark obtained by the output-oriented super efficiency model, there exists an input excess for C. This is reflected by $\hat{\theta}^* = \frac{1}{2}$. That is, unit C can achieve more efficiency if it decreases its input (moves leftward). If unit C increases its use of inputs to infinity, $\hat{\theta}^*$ would approach 0 and its efficiency would decrease as expected but remain greater than 1 as well because it is above the dashed line. If unit C decreases its use of inputs, $\hat{\theta}^*$ would increase and its efficiency would increase as expected.

Our composite score is the same as the score of the model (1) if (1) is feasible and the score will be greater than 1 if (1) is infeasible, which is shown in the following theorem.

Theorem 4. *$\breve{\theta} = \hat{\theta}^*$ if (1) is feasible and $\breve{\theta} > 1$ if (1) is infeasible.*

Proof. If (1) is feasible, the index set R would be empty according to Theorem 1. Therefore, $\breve{\theta} = \hat{\theta}^*$. If (1) is infeasible, the index set R would not be empty according to Theorem 1. Hence $\breve{\theta} = \frac{\sum_{r\in R}\left(\frac{y_r^k}{y_r^k - s_r^* y_r^k}\right)}{|R|} + \hat{\theta}^*$. Since $\frac{\sum_{r\in R}\left(\frac{y_r^k}{y_r^k - s_r^* y_r^k}\right)}{|R|} > 1$, we have $\breve{\theta} > 1$. □

The economic interpretation of our measure is that if a DMU moves rightward or downward, its super efficiency will decrease. So for a DMU to maintain its competitiveness, it had better to move either leftward or upward. When it moves rightward, $\hat{\theta}^*$ will decrease. When it moves downward, $\frac{\sum_{r\in R}\left(\frac{y_{rk}}{y_{rk}-s_r^*}\right)}{|R|}$ will decrease. So $\breve{\theta}$ will decrease when either it moves rightward or downward. Our score gives clues to a DMU when (1) is infeasible that it is outside the efficient boundary and remains competitive because some of its outputs outperform the benchmark. It may further enhance its competitiveness by increasing its outputs (moving upward) or reducing its inputs (moving leftward).

In a similar manner, we can develop our new output-oriented VRS super-efficiency model that is always feasible. The standard output-oriented VRS super-efficiency model can be expressed as

$$
\begin{aligned}
\max\ & \beta \\
\text{s.t.}\ & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j \leqslant x_i^k \quad i = 1,2,\ldots,m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j \geqslant \beta y_r^k \quad r = 1,2,\ldots,s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 \\
& \lambda_j \geqslant 0, \quad j \neq k
\end{aligned}
\tag{4}
$$

We first solve the following linear programming problem which seeks to determine potential input savings $(t_i^* x_i^k)$ in the efficient DMU_k compared to the frontier formed by the rest of DMUs:

$$
\begin{aligned}
\min\ & \sum_{i=1}^{m} t_i \\
\text{s.t.}\ & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j - t_i x_i^k \leqslant x_i^k \quad i = 1,2,\ldots,m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 \\
& \lambda_j \geqslant 0, \quad j \neq k \\
& t_i \geqslant 0, \quad i = 1,2,\ldots,m
\end{aligned}
\tag{5}
$$

Let t_i^* be a set of optimal solution in model (5). Model (4) is feasible if and only if $t_i^* = 0$ for $i = 1,\ldots,m$. We then establish the following new output-oriented VRS super-efficiency model

$$
\begin{aligned}
\max\ & \hat{\beta} \\
\text{s.t.}\ & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_i^j - t_i^* x_i^k \leqslant x_i^k \quad i = 1,2,\ldots,m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_r^j \geqslant \hat{\beta} y_r^k \quad r = 1,2,\ldots,s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 \\
& \lambda_j \geqslant 0, \quad j \neq k
\end{aligned}
\tag{6}
$$

If model (4) is feasible, then $\beta^* = \hat{\beta}^*$. We can also prove that model (6) is always feasible.

It is likely that when infeasibility occurs in model (4), the output-oriented super-efficiency score from model (6) is greater than one, indicating that output super-efficiency does not exist. Consider DMU A in Fig. 1.

The output oriented standard super-efficiency model (4) is infeasible for DMU A. We have $t^* = 1$ in model (5) and $\hat{\beta}^* = 3(>1)$ in model (6). This is because DMU A is projected onto DMU B, and DMU A has to increase both its input and output to reach DMU B. In other words, DMU A has super efficiency in input, but not output.

To address this problem, we can define a new output-oriented super-efficiency score $\breve{\beta}$ in the following manner.

$$
\frac{1}{\breve{\beta}} = \begin{cases} \dfrac{\sum_{i\in I}\left(\frac{x_i^k + t_i^* x_i^k}{x_i^k}\right)}{|I|} + \frac{1}{\hat{\beta}^*}, & \text{if } I \neq \phi. \\ \frac{1}{\hat{\beta}^*}, & \text{if } I = \phi \end{cases}
$$

where $I = \{i | t_i^* > 0\}$ based upon model (5).

We have $\breve{\beta} = \frac{3}{7} < 1$. In fact, we can prove

Theorem 5. *$\breve{\beta} = \hat{\beta}^*$ if (4) is feasible and $\breve{\beta} < 1$ if (4) is infeasible.*

4. Illustration

In this section, we apply our approach to two data sets used in Chen (2004) and Cook et al. (2009). One consists of the 20 largest Japanese companies in 1999 (see Table 1). The other consists of 15 of Fortune's top US cities in 1996 (see Table 2).

For the Japanese companies, the DEA inputs are assets (million \$), equity (million \$) and number of employees and the DEA output is revenue (million \$). Either model (1) or model (4) indicates that 5 of them are VRS-efficient (see last two columns in Table 1). DMU1 is infeasible under input-oriented model (1) and DMU18 is infeasible under output-oriented model (4).

Model (2) shows $s_1^* y_1^1 = 609.1$ (output surplus) ($s_1^* = 0.005704$) for DMU1 under input-orientation, indicating that model (1) is infeasible. The newly developed super-efficiency model (3) yields a score of 1.010356 for DMU1. This further indicates that although DMU1 is infeasible under model (1), DMU1 exhibits super-efficiency in both its inputs and outputs. If we apply the modified super-efficiency score to DMU1, we have $\breve{\theta} = 2.004653$.

We now turn to the output-orientation. For DMU 18, model (5) shows $t_1^* x_1^{18} = 34736.5 (t_1^* = 2.080528)$, $t_2^* x_2^{18} = 1657.7$ ($t_2^* = 2.451856$), and $t_3^* x_3^{18} = 2121 (t_3^* = 0.58046)$ (input savings), indicating that model (4) is infeasible. Model (6) yields a score of 3.515413, indicating that DMU 18 does not have super-efficiency in its output and has super-efficiency in inputs only when infeasibility occurs. If we apply $\breve{\beta}$ to DMU 18, we get a modified output-oriented super-efficiency score of 0.334589.

The data set for the 15 US cities has three inputs, namely, high-end housing price (1,000 US \$), lower-end housing monthly rental (US \$), and number of violent crimes, and three outputs, namely, median household income (US \$), number of bachelor's degrees (million) held by persons in the population, and number of doctors (thousand).

The last two columns of Table 2 report the super-efficiency scores from models (1) and (4). It can be seen that 10 cities are efficient. There are seven infeasibility cases. Table 3 reports the results from our proposed approach.

The results in Table 3 indicate that all cities having no feasible solutions in model (1) or (4) have super-efficiency in both inputs and outputs, as indicated by $\hat{\theta}^* > 1$ and $\beta^* < 1$. This conclusion is

Table 1
Japanese companies.

DMU	Company	Asset	Equity	Employee	Revenue	Model (1) θ^*	Model (4) β^*
1	MITSUI & CO.	50905.3	5137.9	40,000	106793.2	**Infeasible**	**0.98500**
2	ITOCHU CORP.	51432.5	2333.8	5775	106184.1	**6.69295**	**0.75890**
3	MITSUBISHI CORP.	67553.2	7253.2	36,000	104656.3	0.74248	1.01970
4	TOYOTA MOTOR CORP.	112698.1	47,177	183,879	97387.6	0.4108	1.09660
5	MARUBENI CORP.	49742.9	2704.3	5844	91361.7	0.91739	1.12199
6	SUMITOMO CORP.	41168.4	4351.5	30,700	86,921	**1.02091**	**0.97780**
7	NIPPON TELEGRAPH & TEL.	133008.8	47467.1	138,150	74323.4	0.26865	1.43691
8	NISSHO IWAI CORP.	35581.9	1274.4	19,461	66,144	**1.14580**	**0.87120**
9	HITACHI LTD.	73,917	21914.2	328,351	60937.9	0.40528	1.75251
10	MATSUSHITA ELECTRIC INDL.	60,639	26988.4	282,153	58361.6	0.47569	1.82989
11	SONY CORP.	48117.4	13930.7	177,000	51,903	0.54156	1.94791
12	NISSAN MOTOR	52842.1	9583.6	39,467	50263.5	0.47975	2.12450
13	HONDA MOTOR	38455.8	13473.8	112,200	47597.9	0.62931	1.69411
14	TOSHIBA CORP.	46,013	8023.3	198,000	40492.7	0.45933	2.39080
15	FUJITSU LTD.	39052.2	8901.6	188,000	40050.3	0.53631	2.04780
16	TOKYO ELECTRIC POWER	110055.8	12157.7	50,558	38869.5	0.18567	2.74748
17	NEC CORP.	38,015	6517.4	157,773	36356.4	0.50901	2.18981
18	TOMEN CORP.	16,696	676.1	3654	30205.3	**2.89988**	**Infeasible**
19	JAPAN TOBACCO	17023.6	10816.6	31,000	29612.2	0.98076	1.04570
20	MITSUBISHI ELECTRIC CORP.	31,997	4129.6	116,479	28982.2	0.5218	2.26572

Table 2
US cities.

DMU	City	Houseprice	Rental	Violent	Income	B. Degree	Doctor	Model (1) θ^*	Model (4) β^*
1	Seattle	586	581	1193.06	46,928	0.6534	9.878	**1.44335**	**0.91458**
2	Denver	475	558	1131.64	42,879	0.5529	5.301	**1.01593**	**0.94994**
3	Philadelphia	201	600	3468	43,576	1.135	18.2	**Infeasible**	**Infeasible**
4	Minneapolis	299	609	1340.55	45,673	0.729	7.209	**1.22752**	**0.92081**
5	Raleigh	318	613	634.7	40,990	0.319	4.94	**1.16766**	**Infeasible**
6	StLouis	265	558	657.5	39,079	0.515	8.5	**1.51628**	**Infeasible**
7	Cincinnati	467	580	882.4	38,455	0.3184	4.48	0.94968	1.11483
8	Washington	583	625	3286.7	54,291	1.7158	15.41	**Infeasible**	**0.65172**
9	Pittsburgh	347	535	917.04	34,534	0.4512	8.784	**1.04529**	**Infeasible**
10	Dallas	296	650	3714.3	41,984	1.2195	8.82	0.92652	1.04910
11	Atlanta	600	740	2963.1	43,249	0.9205	7.805	0.77243	1.22895
12	Baltimore	575	775	3240.75	43,291	0.5825	10.05	0.73827	1.24860
13	Boston	351	888	2197.12	46,444	1.04	18.208	**Infeasible**	**0.75867**
14	Milwaukee	283	727	778.35	41,841	0.321	4.665	**1.06559**	**0.97314**
15	Nashville	431	695	1245.75	40,221	0.2365	3.575	0.80117	1.14548

Table 3
Super-efficiency for US cities.

		Input-oriented			Output-oriented		
DMU	City	Model (3), $\hat{\theta}^*$	Model (2), Output surplus	Adjusted, $\breve{\theta}$	Model (6), $\hat{\beta}^*$	Model (5), Input saving	Adjusted, $\breve{\beta}$
3	Philadelphia	1.908523	$s_3^* = 0.021172,\ s_3^* y_3^3 = 0.385326$	2.930153	0.453744	$t_1^* = 0.318408,\ t_1^* x_1^3 = 64$	0.283906
5	Raleigh	1.167662		1.167662	0.953379	$t_3^* = 0.035922,\ t_3^* x_3^5 = 22.8$	0.479657
6	StLouis	1.516279		1.516279	0.59373	$t_1^* = 0.196447,\ t_1^* x_1^6 = 52.05848,$ $t_2^* = 0.098379,\ t_2^* x_2^6 = 54.89539$	0.353147
8	Washington	1.077451	$s_1^* = 0.206084,\ s_1^* y_1^8 = 11188.53,$ $s_2^* = 0.323853,\ s_2^* y_2^8 = 0.555666,$	2.446725	0.65171		0.65171
9	Pittsburgh	1.045289		1.045289	0.967668	$t_2^* = 0.042991,\ t_2^* x_2^9 = 23$	0.481602
13	Boston	1.556343	$s_3^* = 0.041453,\ s_3^* y_3^{13} = 0.754777$	2.599589	0.758664		0.758664

* If model (1) (or (4)) is feasible, we do not report results from model (2) (or (5)).

consistent with the results in Cook et al. (2009). Table 3 also reports our modified super-efficiency scores when infeasibility occurs.

5. Conclusions

The current paper extends Chen (2005) and Cook et al. (2009) by providing an approach for addressing the infeasibility issue in the super-efficiency DEA models. Our approach can detect whether a VRS super-efficiency model is infeasible and the input savings (output surpluses) of a particular DMU under evaluation. Our numerical examples show that infeasibility may imply that a DMU does not exhibit super-efficiency in inputs or outputs, although sometimes infeasibility indicates super-efficiency in both inputs and outputs. Our approach is closely related to Cook et al. (2009) which is designed for DMUs with infeasible solutions. Our approach is applicable to all DMUs and yields results identical to the standard VRS super-efficiency model when infeasibility does not exist. The current study also extends Cook et al. (2009) by fully incorporating the input saving in all inputs and output surplus in all outputs.

References

Andersen, P., Petersen, N.C., 1993. A procedure for ranking efficient units in data envelopment analysis. Management Science 39, 1261–1264.

Charnes, A., Cooper, W.W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European Journal of Operational Research 2, 429–444.

Chen, Y., 2004. Ranking efficient units in DEA. OMEGA 32, 213–219.

Chen, Y., 2005. Measuring super-efficiency in DEA in the presence of infeasibility. European Journal of Operational Research 161, 545–551.

Cook, W.D., Liang, L., Zha, Y., Zhu, J., 2009. A Modified Super-efficiency DEA Model for Infeasibility. Journal of Operational Research Society 69, 276–281.

Fukuyama, H., Weber, W.L., 2009. A directional slacks-based measure of technical inefficiency. Socio-Economic Planning Sciences 43, 274–287.

Lovell, C.A.K., Rouse, A.P.B., 2003. Equivalent standard DEA models to provide super-efficiency scores. Journal of the Operational Research Society 54 (1), 101–108.

Portela, M., Thanassoulis, E., Simpson, G., 2004. Negative data in DEA: A directional distance approach applied to bank branches. Journal of the Operational Research Society 55 (10), 1111–1121.

Russel, R.R., 1985. Measures of technical efficiency. Journal of Economic Theory 35, 109–126.

Russel, R.R., 1988. On the axiomatic approach to the measurement of technical efficiency. In: Eichhorn, W. (Ed.), Measurement in Economics. Physica, Heidelberg, pp. 207–217.

Seiford, L.M., Zhu, J., 1999. Infeasibility of super-efficiency data envelopment analysis models. INFOR 37 (May), 174–187.

Tone, K., 2001. A slacks-based measure of efficiency in data envelopment analysis. European Journal of Operational Research 130, 498–509.