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Interfaces with Other Disciplines

Network DEA: Additive efficiency decomposition

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ABSTRACT

In conventional DEA analysis, DMUs are generally treated as a black-box in the sense that internal structures are ignored, and the performance of a DMU is assumed to be a function of a set of chosen inputs and outputs. A significant body of work has been directed at problem settings where the DMU is characterized by a multistage process; supply chains and many manufacturing processes take this form. Recent DEA literature on serial processes has tended to concentrate on closed systems, that is, where the outputs from one stage become the inputs to the next stage, and where no other inputs enter the process at any intermediate stage. The current paper examines the more general problem of an open multistage process. Here, some outputs from a given stage may leave the system while others become inputs to the next stage. As well, new inputs can enter at any stage. We then extend the methodology to examine general network structures. We represent the overall efficiency of such a structure as an additive weighted average of the efficiencies of the individual components or stages that make up that structure. The model therefore allows one to evaluate not only the overall performance of the network, but as well represent how that performance decomposes into measures for the individual components of the network. We illustrate the model using two data sets.

1. Introduction

Data envelopment analysis (DEA) is a tool for measuring the relative efficiency of peer decision making units (DMUs) that have multiple inputs and outputs. As indicated in Cooper et al. (2004), the definition of a DMU is generic and flexible. DMUs can be any entities engaged in many different activities in many different contexts, e.g., hospitals, US Air Force wings, universities, cities, courts, business firms, and others. In conventional DEA, DMUs are treated as a black-box in the sense that internal structures are generally ignored, and the performance of a DMU is assumed to be a function of the chosen inputs and outputs. In many cases, DMUs may have internal or network structures; see for example, Färe and Grosskopf (1996), Castelli et al. (2004), and Tone and Tsutsui (2009). In the latter case, the authors provide a slacks-based model that captures the overall efficiency of the DMU, and provides, as well, measures for the components (referred to as divisions) or stages that make up the DMU. The overall efficiency is expressed as a weighted average of the component efficiencies, where weights are exogenously imposed to reflect the perceived importance of the components. The current paper focuses on the derivation of a radial measure of efficiency that can be decomposed into a convex combination of radial measures for the individual components that make up the DMU. In our case the weights are variables, and not imposed exogenously.

These types of DMUs have not only inputs and outputs, but also intermediate measures that flow from one stage to the other. Each stage may also have its own inputs and outputs. Recently, a number of studies have focused on DMUs that appear as two-stage processes. For example, Seiford and Zhu (1999) view the profitability and marketability of US commercial banks as a two-stage process. In their study, profitability is measured in the first stage using labor and assets as inputs and profits and revenues as outputs. In the second stage for marketability, the profits and revenue are then used as inputs, while market value, returns and earnings per share, constitute the outputs. Kao and Hwang (2008) describe a two-stage process where 24 non-life insurance companies use operating and insurance expenses to generate premiums in the first stage, and then underwriting and investment profits in the second stage. Other examples include the impact of information technology use on bank branch performance (Chen and Zhu, 2004), two stage Major League Baseball performance (Sexton and Lewis, 2003), health care application (Chilingerian and Sherman, 2004), and many others.

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Some of the above studies use the conventional DEA approach; see, for example, Seiford and Zhu (1999), and Sexton and Lewis (2003). This approach does not, however, address potential conflicts between the two stages arising from the intermediate measures. For example, the second stage may have to reduce its inputs (intermediate measures) in order to achieve an efficient status. Such an action would, however, imply a reduction in the first stage outputs, thereby reducing the efficiency of that stage.

Novel approaches have been developed to model the intermediate measures that exist between stages within DMUs. For example, Kao and Hwang (2008) modify the standard radial DEA model by decomposing the overall efficiency of the DMU into the product of the efficiencies of the two stages. Such *multiplicative* efficiency decomposition is also studied in Liang et al. (2008), where three DEA models/efficiency decompositions are developed using game theory concepts. More recently, Chen et al. (2009) present a methodology for representing overall radial efficiency of a DMU as an *additive* weighted average of the radial efficiencies of the individual stages or components that make up the DMU.

While the approaches of Kao and Hwang (2008), Liang et al. (2008), and Chen et al. (2009) can be extended to DMUs that have more than two stages, such an extension requires that the multistage processes share the unique feature that all outputs from any stage represent the only inputs to the next stage. In other words, except for the first stage, all other stages do not have their own independent inputs (and/or outputs), that enter (exit) the process at that point. While these *closed* systems do exist, the more prevalent case is that where each stage is *open*, that is it has its own inputs (and/or outputs) in addition to the intermediate measures.

Such open multistage structures are relatively common, particularly in processing industries. Consider, for example, the situation in which a coal mining company wishes to evaluate the efficiency of a set of collieries (mining operations) in a large coal field. Typically, the process of delivering finished products to the customer is multistage in nature. In crude terms, *stage 1* would involve the *extraction* of the raw or run-of-mine (ROM) coal from underground or open pit coal reserves. At the mine site, the ROM is generally put through a process where screens separate the product into different size categories; e.g., a 'more than one inch in diameter' category, and a 'less than one inch' category. The resulting 'size grades', representing the outputs from this first stage, are then transported to an on-site *washing facility*, which might be deemed *stage 2*. The washing process filters out any material below a certain specific gravity; this portion is unsuitable for sale and is discarded. A portion of the remaining usable coal (outputs from stage 2) is sold to the open market as a finished product, and at management's discretion (based on estimates of the demand), the remaining product is sent to *stage 3*, the *crusher*. The crushing process also produces waste or discard, with the remaining material, sometimes referred to as 'middlings', being sold or blended with other materials to make such products as briquettes. This latter process might be thought of as *stage 4*.

Numerous such examples from processing industries exist. In many cases a portion of the outputs from one stage may be in 'finished' form and go to the consumer market, with the remainder being reprocessed at the next stage to get a more pure form of the product. The petrochemical industry, perfume manufacturing and so on, are examples.

It is important to note that the models of Kao and Hwang (2008), Liang et al. (2008) and Chen et al. (2009) concentrate specifically on pure serial processes. The current paper develops linear models for DMUs that have multiple stages, with each stage being open, having its own inputs and outputs. We also obtain an additive efficiency decomposition of the overall efficiency score. The advantage of additive efficiency decomposition is that we can also study performance under assumptions of both constant returns to scale (CRS) and variable returns to scale (VRS). As noted above, we adopt a radial efficiency framework, as compared to the slacks-based framework of Tone and Tsutsui (2009).

For ease of notation, we begin in Section 2 by examining open serial systems as described above. We present a model for measuring the overall radial efficiency of the general serial multistage process, and show that this measure can be decomposed into radial measures of efficiency for the components or stages making up the overall process. Section 3 then extends this model structure to include more complex multistage processes. Our approach is illustrated in Section 4 with the supply chain data set in Liang et al. (2006). As well, we re-evaluate the data set provided in Tone and Tsutsui (2009).

2. DEA model for general multistage serial processes via additive efficiency decomposition

Consider the P-stage process pictured in Fig. 1. We denote the input vector to stage 1 by z_o. The output vectors from stage p ($p = 1,\ldots,P$) take two forms, namely z_p^1 and z_p^2. Here, z_p^1 represents that output that leaves the process at this stage and is not passed on as input to the next stage. The vector z_p^2 represents the amount of output that becomes input to the next ($p+1$) stage. These types of intermediate measures are called *links* in Tone and Tsutsui (2009). In addition, there is the provision for new inputs z_p^3 to enter the process at the beginning of stage $p+1$. Specifically, when $p = 2,3,\ldots$, we define:

(1) z_{pr}^{j1} the rth component ($r = 1,\ldots,R_p$) of the R_p-dimensional **output** vector for DMU$_j$ flowing from stage p, that *leaves* the process at that stage, and is not passed on as an input to stage $p+1$.
(2) z_{pk}^{j2} the kth component ($k = 1,\ldots,S_p$) of the S_p-dimensional **output** vector for DMU$_j$ flowing from stage p, and is passed on as a portion of the **inputs** to stage $p+1$.
(3) z_{pi}^{j3} the ith component ($i = 1,\ldots,I_p$) of the I_p-dimensional **input** vector for DMU$_j$ at the stage $p+1$, that enters the process at the beginning of that stage.

Note that in the last stage P, all the outputs are viewed as z_{pr}^{j1}, as they leave the process.

We denote the multipliers (weights) for the above factors as:

(1) u_{pr} is the multiplier for the output component z_{pr}^{j1} flowing from stage p.
(2) η_{pk} is the multiplier for the output component z_{pk}^{j2} at stage p, and is as well the multiplier for that same component as it becomes an input to stage $p+1$.
(3) v_{pi} is the multiplier for the input component z_{pi}^{j3} entering the process at the beginning of stage $p+1$.

Thus, when $p = 2,3,\ldots$, the efficiency ratio for DMU$_j$ (for a given set of multipliers) would be expressed as:

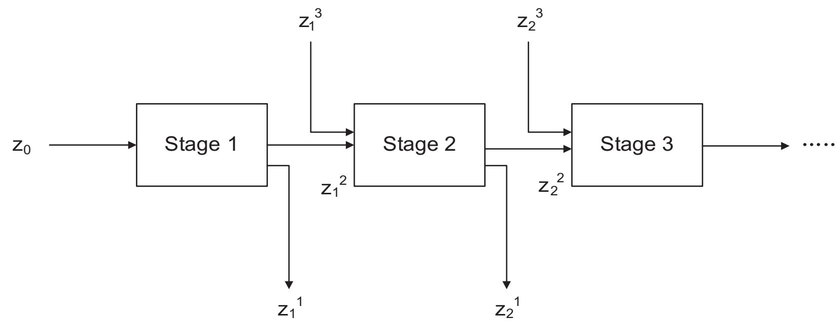

Fig. 1. Serial multistage DMU.

$$\theta_p = \left(\sum_{r=1}^{R_p} u_{pr} z_{pr}^{j1} + \sum_{k=1}^{S_p} \eta_{pk} z_{pk}^{j2}\right) \Bigg/ \left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{j3}\right). \tag{1}$$

Note that there are no outputs flowing into stage 1. The efficiency measure for stage 1 of the process (namely, $p = 1$), for DMU_j becomes

$$\theta_1 = \left(\sum_{r=1}^{R_1} u_{1r} z_{1r}^{j1} + \sum_{k=1}^{S_1} \eta_{1k} z_{1k}^{j2}\right) \Bigg/ \sum_{i=1}^{I_0} v_{0i} z_{0i}^{j}, \tag{2}$$

where z_{0i}^j are the (only) inputs to the first stage represented by the input vector z_o.

We claim that the overall efficiency measure of the multistage process can reasonably be represented as a convex linear combination of the P (stage–level) measures, namely

$$\theta = \sum_{p=1}^{P} w_p \theta_p \quad \text{where} \ \sum_{p=1}^{P} w_p = 1.$$

Note that the weights w_p are intended to represent the relative importance or contribution of the performances of individual stages p to the overall performance of the entire process. One reasonable choice for weights w_p is the proportion of total resources for the process that are devoted to stage p, and reflecting the relative size of that stage. To be more specific, $\sum_{i=1}^{I_0} v_{0i} z_{0i}^j + \sum_{p=2}^{P}\left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{j3}\right)$ represents the total size of or total amount of resources consumed by the entire process, and we define the w_p to be the proportion of the total input used at the pth stage. We then have

$$w_1 = \sum_{i=1}^{I_0} v_{0i} z_{0i}^j \Bigg/ \left\{\sum_{i=1}^{I_0} v_{0i} z_{0i}^j + \sum_{p=2}^{P}\left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{j3}\right)\right\},$$

$$w_p = \left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{j3}\right) \Bigg/ \left\{\sum_{i=1}^{I_0} v_{0i} z_{0i}^j + \sum_{p=2}^{P}\left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{j3}\right)\right\}, \quad p > 1.$$

Thus, we can write the overall efficiency θ in the form

$$\theta = \sum_{p=1}^{P}\left(\sum_{r=1}^{R_p} u_{pr} z_{pr}^{j1} + \sum_{k=1}^{S_p} \eta_{pk} z_{pk}^{j2}\right) \Bigg/ \left\{\sum_{i=1}^{I_0} v_{0i} z_{0i}^j + \sum_{p=2}^{P}\left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{j3}\right)\right\}, \tag{3}$$

We then set out to optimize the overall efficiency θ of the multistage process, subject to the restrictions that the individual measures θ_p must not exceed unity, or in the linear programming format, after making the usual Charnes and Cooper transformation,

$$\begin{aligned}
\max \quad & \sum_{p=1}^{P}\left(\sum_{r=1}^{R_p} u_{pr} z_{pr}^{o1} + \sum_{k=1}^{S_p} \eta_{pk} z_{pk}^{o2}\right), \\
\text{subject to} \quad & \left\{\sum_{i=1}^{I_0} v_{0i} z_{0i}^{o} + \sum_{p=2}^{P}\left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{o2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{o3}\right)\right\} = 1, \\
& \left(\sum_{r=1}^{R_1} u_{1r} z_{1r}^{j1} + \sum_{k=1}^{S_1} \eta_{1k} z_{1k}^{j2}\right) \leqslant \sum_{i=1}^{I_0} v_{0i} z_{0i}^{j}, \\
& \left(\sum_{r=1}^{R_p} u_{pr} z_{pr}^{j1} + \sum_{k=1}^{S_p} \eta_{pk} z_{pk}^{j2}\right) \leqslant \left(\sum_{k=1}^{S_{p-1}} \eta_{p-1k} z_{p-1k}^{j2} + \sum_{i=1}^{I_p} v_{p-1i} z_{p-1i}^{j3}\right) \forall j, \\
& u_{pr}, \quad \eta_{pk}, \quad v_{pi}, \quad v_{0i} > 0.
\end{aligned} \tag{4}$$

Note that we should impose the restriction that the overall efficiency scores for each j should not exceed unity, but since these are redundant, this is unnecessary.

Note again that the w_p, as defined above, are variables related to the inputs and the intermediate measures. By virtue of the optimization process, it can turn out that some $w_p = 0$ at optimality. To overcome this problem, one can impose bounding restrictions $w_p \geqslant \beta$, where β is a selected constant. This is illustrated in the examples of Section 4.

3. General multistage processes

In the process discussed in the previous section it is assumed that the components of a DMU are arranged in series as depicted in Fig. 1. There, at each stage p, the inputs took one of two forms, namely (1) those that are outputs from the previous stage $p-1$, and (2) new inputs that enter the process at the start of stage p. On the output side, those (outputs) emanating from stage p take two forms as well, namely (1) those that leave the system as finished 'products', and (2) those that are passed on as inputs to the *immediate* next stage $p+1$.

The model presented to handle such strict serial processes is easily adapted to more general network structures. Specifically, the efficiency ratio for an overall process can be expressed as the weighted average of the efficiencies of the individual components. The efficiency of any given component is the ratio of the total output to the total input corresponding to that component. Again, the weight w_p to be applied to any component p is expressed as

$$w_p = (\text{component } p \text{ input})/(\text{total input across all components}).$$

There is no convenient way to represent a network structure that would lend itself to a generic mathematical representation analogous to model (4) above. The sequencing of activities and the source of inputs and outputs for any given component will differ from one type of process to another. However, as a simple illustration, consider the following two examples of network structures:

3.1. Parallel processes

Consider the process depicted in Fig. 2. Here, an initial input vector z_o enters component 1. Three output vectors exit this component, that is z_1^1 leaves the process, z_1^2 is passed on as an input to component 2, and z_1^3 as an input to component 3. Additional inputs z_1^4 and z_1^5 enter components 2 and 3 respectively, from outside the process. Components 2 and 3 have z_2^1 and z_3^1, respectively as output vectors which are passed on as inputs to component 4, where a final output vector z_4^1 is the result.

3.1.1. Component efficiencies

Component 1 efficiency ratio: $\theta_1 = (u_1 z_1^1 + \eta_1^2 z_1^2 + \eta_1^3 z_1^3)/v_o z_o$
Component 2 efficiency ratio: $\theta_2 = \eta_2^1 z_2^1/(\eta_1^2 z_1^2 + v_1 z_1^4)$
Component 3 efficiency ratio: $\theta_3 = \eta_3^1 z_3^1/(\eta_1^3 z_1^3 + v_2 z_1^5)$
Component 4 efficiency ratio: $\theta_4 = u_4 z_4^1/(\eta_2^1 z_2^1 + \eta_3^1 z_3^1)$

3.1.2. Component weights

Note that the total (weighted) input across all components is given by the sum of the denominators of θ_1 through θ_4, namely

$$I = v_o z_o + \eta_1^2 z_1^2 + v_1 z_1^4 + \eta_1^3 z_1^3 + v_2 z_1^5 + \eta_2^1 z_2^1 + \eta_3^1 z_3^1.$$

Now express the w_p as:

$$w_1 = v_o z_o/I,$$
$$w_2 = (\eta_1^2 z_1^2 + v_1 z_1^4)/I,$$
$$w_3 = (\eta_1^3 z_1^3 + v_2 z_1^5)/I,$$
$$w_4 = (\eta_2^1 z_2^1 + \eta_3^1 z_3^1)/I.$$

With this, the overall network efficiency ratio is given by

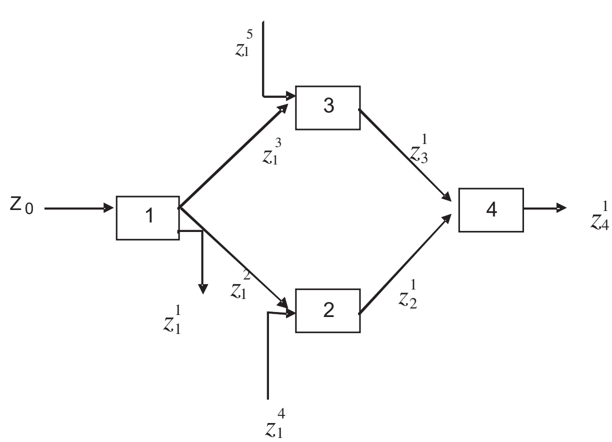

Fig. 2. Multistage DMU with parallel processes.

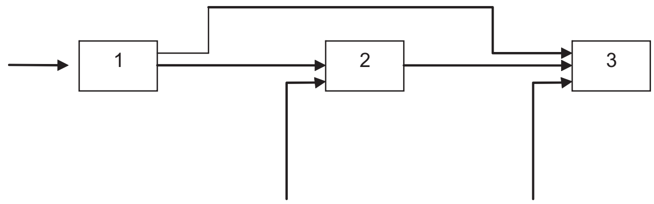

Fig. 3. Non-immediate successor flows.

$$\theta = \sum_{p=1}^{4} w_p \theta_p = \left(u_1 z_1^1 + \eta_1^2 z_1^2 + \eta_1^3 z_1^3 + \eta_2^1 z_2^1 + \eta_3^1 z_3^1 + u_4 z_4^1\right)/I.$$

And one then proceeds, as in (4) above, to derive the efficiency of each DMU and its components.

3.2. Non-immediate successor flows

In the previous example all flows of outputs from a stage or component either leave the process entirely or enter as an input to an *immediate successor* stage. In Fig. 1, stage p outputs flow to stage $p+1$. In Fig. 2, the same is true except that there is more than one immediate successor of stage 1.

Consider Fig. 3. Here, the inputs to stage 3 are of three types, namely outputs from stage 2, inputs coming from outside the process, and outputs from a previous, but not immediately previous stage. Again the above rationale for deriving weights w_p can be applied and a model equivalent to (4) solved to determine the decomposition of an overall efficiency score into scores for each of the components in the process.

4. An illustrative application

We here re-visit the supply chain data set used in Liang et al. (2006). This data set consists of a two-stage process, or a seller-buyer supply chain. The inputs to the first stage (seller) are labor $\left(z_{01}^j\right)$, operating cost $\left(z_{02}^j\right)$ and shipping cost $\left(z_{03}^j\right)$. The outputs from the first stage are number of product A shipped $\left(z_{11}^{j2}\right)$, number of product B shipped $\left(z_{12}^{j2}\right)$ and number of product C shipped $\left(z_{13}^{j2}\right)$. This data set assumes that all outputs from the first stage become inputs to the second stage, i.e., there is no z_1^1. There is one input to the second stage (buyer), labor $\left(z_{11}^{j3}\right)$, and two outputs from the second stage, sales $\left(z_{21}^{j1}\right)$ and profits $\left(z_{22}^{j1}\right)$. Table 1 provides the data set.

In this case, we have, for DMU$_o$

$$w_1 = \sum_{i=1}^{3} v_{0i} z_{0i}^o \Bigg/ \left(\sum_{i=1}^{3} v_{0i} z_{0i}^o + \sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3}\right),$$

$$w_2 = \left(\sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3}\right) \Bigg/ \left(\sum_{i=1}^{3} v_{0i} z_{0i}^o + \sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3}\right),$$

$$\begin{aligned}
\max \quad & \sum_{k=1}^{3} \eta_{2k} z_{1k}^{o2} + \sum_{r=1}^{2} u_{2r} z_{2r}^{o1}, \\
\text{subject to} \quad & \sum_{i=1}^{3} v_{0i} z_{0i}^o + \sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3} = 1, \\
& \sum_{k=1}^{3} \eta_{1k} z_{1k}^{j2} \leqslant \sum_{i=1}^{3} v_{0i} z_{0i}^j, \quad j = 1, \ldots, 10 \text{ (for stage 1)}, \\
& \sum_{r=1}^{2} u_{2r} z_{2r}^{j1} \leqslant \sum_{k=1}^{3} \eta_{1k} z_{1k}^{j2} + v_{11} z_{11}^{j3}, \quad j = 1, \ldots, 10 \text{ (for stage 2)},
\end{aligned} \tag{5}$$

where efficiency scores for DMU$_o$ in stages 1 and 2 can be expressed as

$$\theta_1 = \sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} \Bigg/ \sum_{i=1}^{3} v_{0i} z_{0i}^o,$$

$$\theta_2 = \sum_{r=1}^{2} u_{2r} z_{2r}^{o1} \Bigg/ \left(\sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3}\right).$$

Table 2 reports the results from model (5) where the last two columns display the efficiency scores derived from the cooperative model of Liang et al. (2006). Note that the differences between the two approaches are not significant. For example, the two approaches yield identical efficiency scores for the two stages for DMUs, 2, 5, 6, and 9. Liang et al.'s (2006) approach is based upon a non-linear program and its solution is obtained by using heuristic search. While the current approach uses a linear program and guarantees a global optimal solution.

Table 1
Data set.

DMU	Labor z_{01}^{j}	Operating cost z_{02}^{j}	Shipping cost z_{03}^{j}	Product A z_{11}^{j2}	Product B z_{12}^{j2}	Product C z_{13}^{j2}	Labor z_{11}^{j3}	Sales z_{21}^{j1}	Profits z_{22}^{j1}
1	9	50	1	20	10	5	8	100	25
2	10	18	10	10	15	7	10	70	20
3	9	30	3	8	20	2	8	96	30
4	8	25	1	20	20	10	10	80	20
5	10	40	5	15	20	5	15	85	15
6	7	35	2	35	10	5	5	90	35
7	7	30	3	10	25	8	10	100	30
8	12	40	4	20	25	4	8	120	10
9	9	25	2	10	10	5	15	110	15
10	10	50	1	20	15	9	10	80	20

Table 2
Results.

DMU	Our results (Model (5))					Liang et al. (2006)	
	Overall score	w_1	w_2	θ_1	θ_2	θ_1	θ_2
1	0.92495	0.30843	0.69157	0.75666	1	1	0.89394
2	0.86486	0.51974	0.48026	0.92403	0.80082	0.92403	0.80082
3	0.85898	0.34817	0.65183	0.59497	1	0.69106	1
4	0.77381	0.5	0.5	1	0.54762	1	0.62786
5	0.62073	0.46194	0.53806	0.67595	0.57332	0.67595	0.57332
6	1	0.27992	0.72008	1	1	1	1
7	0.90405	0.5	0.5	1	0.80811	1	0.81888
8	0.92886	0.21477	0.78523	0.66875	1	0.74667	1
9	0.78091	0.43817	0.56183	0.5	1	0.5	1
10	0.75444	0.54281	0.45719	0.84226	0.65018	1	0.59596

Note that the average of the two stages' efficiency scores is used as the objective function in Liang et al.'s (2006) non-linear model, namely, the weights for the two individual efficiency scores are equal, $w_1 = w_2$. The current approach yields $w_1 = w_2 = 0.5$ for DMUs 4 and 7. Yet, our results are different from those obtained from Liang et al. (2006). For example, in DMU 7, the efficiency score for the second stage is 0.54762 compared to 0.81888 from Liang et al. (2006). This is due to the fact that our choice of weights actually introduces some sort of value judgment into the DEA model, and restricts the multiplier values in model (5). This is why Liang et al.'s (2006) score is larger than ours when $w_1 = w_2 = 0.5$ in optimality.

Note that weights w_p $(p = 1, 2, \ldots, P)$ defined in our paper are actually variables related to the multiplier decision variables. We next, therefore, impose additional restrictions on w_1 and w_2 in model (5) via

$$w_1 = \left\{ \sum_{i=1}^{3} v_{0i} z_{0i}^{o} \middle/ \left(\sum_{i=1}^{3} v_{0i} z_{0i}^{o} + \sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3} \right) \right\} \geqslant \beta_1,$$

$$w_2 = \left\{ \left(\sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3} \right) \middle/ \left(\sum_{i=1}^{3} v_{0i} z_{0i}^{o} + \sum_{k=1}^{3} \eta_{1k} z_{1k}^{o2} + v_{11} z_{11}^{o3} \right) \right\} \geqslant \beta_2,$$

where β_1 and β_2 are user-specified parameters. In this way, we can perform sensitivity analysis on w_1 and w_2.

We first impose $\beta_1 = \beta_2$ and change β_1 and β_2 0.1 to 0.5 with a 0.1 increment each time. Note that when $\beta_1 = \beta_2 = 0.5$, we explicitly require that $w_1 = w_2 = 0.5$ as in Liang et al. (2006). Table 3 reports the results when $\beta_1 = \beta_2 = 0.5$. Both our approach and Liang et al.'s (2006) yield identical efficiency scores for DMU 9. Except for DMU 1, Liang et al.'s (2006) score is larger than ours when $w_1 = w_2 = 0.5$ in optimality. For

Table 3
Results with $\beta_1 = 0.5$, $\beta_2 = 0.5$.

DMU	Overall score	w_1	w_2	θ_1	θ_2
1	0.86323	0.5	0.5	0.72645	1
2	0.85303	0.5	0.5	0.9222	0.78386
3	0.83629	0.5	0.5	0.67258	1
4	0.77381	0.5	0.5	1	0.54762
5	0.61749	0.5	0.5	0.67595	0.55903
6	0.99678	0.5	0.5	0.99357	1
7	0.90405	0.5	0.5	1	0.80811
8	0.81756	0.5	0.5	0.72772	0.9074
9	0.75	0.5	0.5	0.5	1
10	0.75435	0.5	0.5	0.85137	0.65732

Table 4
Results with $\beta_1 = \beta_2 = 0.1$ (0.2, 0.3, 0.4).

DMU	Overall score	w_1	w_2	θ_1	θ_2
2	0.86486	0.51974	0.48026	0.92403	0.80082
4	0.77381	0.5	0.5	1	0.54762
5	0.62073	0.46194	0.53806	0.67595	0.57332
6	1	0.31591	0.68409	1	1
7	0.90405	0.5	0.5	1	0.80811
9	0.78091	0.43817	0.56183	0.5	1
10	0.75444	0.54281	0.45719	0.84226	0.65018

Table 5
Results for DMUs 1, 3, and 8.

DMU	Overall score	w_1	w_2	θ_1	θ_2	
1	0.92495	0.30843	0.69157	0.75666	1	$\beta_1 = \beta_2 = 0.1, 0.2, 0.3$
1	0.90182	0.4	0.6	0.75455	1	$\beta_1 = \beta_2 = 0.4$
3	0.85898	0.34817	0.65183	0.59497	1	$\beta_1 = \beta_2 = 0.1, 0.2, 0.3$
3	0.85186	0.4	0.6	0.62966	1	$\beta_1 = \beta_2 = 0.4$
8	0.92886	0.21477	0.78523	0.66875	1	$\beta_1 = \beta_2 = 0.1, 0.2$
8	0.91627	0.3	0.7	0.72091	1	$\beta_1 = \beta_2 = 0.3$
8	0.89238	0.4	0.6	0.73095	1	$\beta_1 = \beta_2 = 0.4$

DMU 1, the definition of our weights and restrictions on our weights turn the efficient stage 1 under Liang et al.'s (2006) approach into an inefficient stage, and the inefficient stage 2 under Liang et al.'s (2006) approach into efficient.

Table 4 reports the results for DMUs 2, 4, 5, 6, 7, 9 and 10 whose efficiency scores along with the optimized weights remain unchanged when $\beta_1 = \beta_2 = 0.1$, 0.2, 0.3 and 0.4, respectively.

Table 5 reports the results for DMUs 1, 3 and 8 whose efficiency scores changed when β_1 and β_2 are changed (see the last column of Table 5). For DMUs 1 and 3, change in the efficiency scores does not occur until $\beta_1 = \beta_2 = 0.4$. For DMU 8, a change in the efficiency score for the first stage is observed when $\beta_1 = \beta_2 = 0.3$ and 0.4.

It can be seen that up to $\beta_1 = \beta_2 = 0.3$, most of the DMUs have the same weights and efficiency scores with respect to different values of β_1 and β_2. As expected, when $\beta_1 = \beta_2 = 0.4$, some of the resulting weights are different from the previous cases. However, we note that the efficiency scores do not change significantly. We also note that the efficiency scores for the second stage do not change when β_1 and β_2 are increased from 0.1 to 0.4.

Table 6
Data set in Tone and Tsutsui (2009).

	Stage 1	Stage 2		Stage 3		Intermediate measure	
	Input 1	Input 2	Output 2	Input 3	Output 3	Link12	Link23
A	0.838	0.277	0.879	0.962	0.337	0.894	0.362
B	1.233	0.132	0.538	0.443	0.18	0.678	0.188
C	0.321	0.045	0.911	0.482	0.198	0.836	0.207
D	1.483	0.111	0.57	0.467	0.491	0.869	0.516
E	1.592	0.208	1.086	1.073	0.372	0.693	0.407
F	0.79	0.139	0.722	0.545	0.253	0.966	0.269
G	0.451	0.075	0.509	0.366	0.241	0.647	0.257
H	0.408	0.074	0.619	0.229	0.097	0.756	0.103
I	1.864	0.061	1.023	0.691	0.38	1.191	0.402
J	1.222	0.149	0.769	0.337	0.178	0.792	0.187

Table 7
Results on three-stage process.

	Overall	Stage 1	Stage 2	Stage 3	w_1	w_2	w_3
A	0.579	0.410	0.646	0.971	0.46	0.41	0.13
B	0.386	0.211	0.339	0.414	0.10	0.10	0.80
C	1.000	1.000	1.000	0.999	0.42	0.48	0.10
D	0.917	0.225	0.942	1.000	0.10	0.10	0.80
E	0.478	0.167	0.501	0.953	0.36	0.42	0.22
F	0.598	0.470	0.656	0.984	0.51	0.37	0.11
G	0.762	0.551	0.717	0.983	0.24	0.44	0.32
H	0.675	0.711	0.599	0.843	0.46	0.44	0.10
I	0.922	0.245	1.000	0.990	0.10	0.64	0.26
J	0.476	0.249	0.423	0.511	0.10	0.10	0.80

We also performed calculations when β_1 is fixed at 0.2 and β_2 is changed from 0.3 to 0.8 with an increment of 0.1 each time (results are not reported here). In overall, the efficiency scores do not change significantly.

The above sensitivity analysis indicates that efficiency scores obtained based upon our approach are robust with respect to our choice of weights.

We finally apply our approach to the numerical example used in Tone and Tsutsui (2009). Table 6 provides the data. We have two intermediate measures or outputs flow from one stage to the other. Table 7 reports the results. In this case, if we do not impose a lower bound for the w_p ($p = 1,2,3$), we have some $w_p = 1$ at optimality (for DMUs B, D, I and J). Therefore, we impose $w_p \geqslant 0.1$ ($p = 1,2,3$) in model (4). Because our approach is different from Tone and Tsutsui's (2009) and our choice of weights introduces restrictions on the multipliers, our results are different from theirs.

5. Conclusions

The current paper develops a DEA approach for DMUs that have a general multistage or network structure. We first examine pure serial networks where each stage has its own inputs and two types of outputs. One type of output from any given stage p is passed on as an input to the next stage, and the other type exits the process at stage p. Work closely related to the current paper is the non-linear approach of Liang et al. (2006) where a two-member supply chain structure is studied. While Liang et al. (2006) developed a heuristic search algorithm after converting the non-linear model into a parametric linear model, their approach cannot be generalized into cases where supply chains have more than two members. Our approach can, however, handle via a linear model, situations where more than two stages are present.

In general, the intermediate measures are those that exist between two members of the network. In many cases, the intermediate measures are obvious, as indicated in our examples mentioned in the Introduction. Tone and Tsutsui (2009) provides other good examples. Sometimes, the selection of intermediate measures is not so obvious. The important thing is that intermediate measures are neither "inputs" (to be reduced) nor "outputs" (to be increased), rather these measures need to be "coordinated" to determine their efficient levels (see, Kao and Hwang (2008) and Liang et al. (2008).)

The current paper develops models under the assumption of constant returns to scale (CRS). We should point out that our models can directly be applied to variable returns to scale (VRS) by adding the free-in-sign variable in our ratio efficiency definition, just as in the standard VRS DEA model. Such an extension is difficult in the approach of Liang et al. (2006), as the resulting model is highly non-linear and cannot be converted into a parametric linear program. Therefore, the current paper extends and generalizes the approach of Liang et al. (2006).

We demonstrate that the approach applied to serial process applies as well to more general network structures. This is illustrated using the data provided in Tone and Tsutsui's (2009).

References

Castelli, L., Pesenti, R., Ukovich, W., 2004. DEA-like models for the efficiency evaluation of hierarchically structured units. European Journal of Operational Research 154 (2), 465–476.
Chen, Y., Zhu, J., 2004. Measuring information technology's indirect impact on firm performance. Information Technology and Management Journal 5 (1–2), 9–22.
Chen, Y., Cook, W.D., Li, N., Zhu, J., 2009. Additive efficiency decomposition in two-stage DEA. European Journal of Operational Research 196, 1170–1176.
Chilingerian, J., Sherman, H.D., 2004. Health care applications: From hospitals to physician, from productive efficiency to quality frontiers. In: Cooper, W.W., Seiford, L.M., Zhu, J. (Eds.), Handbook on Data Envelopment Analysis. Springer, Boston.
Cooper, W.W., Seiford, L.M., Zhu, J., 2004. Data envelopment analysis: History, models and interpretations. In: Cooper, W.W., Seiford, L.M., Zhu, J. (Eds.), Handbook on Data Envelopment Analysis. Kluwer Academic Publishers, Boston, pp. 1–39 (Chapter 1).
Färe, R., Grosskopf, S., 1996. Productivity and intermediate products: A frontier approach. Economics Letters 50, 65–70.
Kao, C., Hwang, S.N., 2008. Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. European Journal of Operational Research 185 (1), 418–429.
Liang, L., Yang, F., Cook, W.D., Zhu, J., 2006. DEA models for supply chain efficiency evaluation. Annals of Operations Research 145 (1), 35–49.
Liang, L., Cook, W.D., Zhu, J., 2008. DEA models for two-stage processes: Game approach and efficiency decomposition. Naval Research Logistics 55, 643–653.
Seiford, L.M., Zhu, J., 1999. Profitability and marketability of the top 55 US commercial banks. Management Science 45 (9), 1270–1288.
Sexton, T.R., Lewis, H.F., 2003. Two-stage DEA: An application to major league baseball. Journal of Productivity Analysis 19 (2–3), 227–249.
Tone, K., Tsutsui, M., 2009. Network DEA: A slacks-based measure approach. European Journal of Operational Research 197 (1), 243–252.