Decomposing technical efficiency and scale elasticity in two-stage network DEA

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A B S T R A C T

The constant returns to scale assumption maintained by neoclassical theorists for justifying the black-box structure of production technology in long run does not necessarily allow one to infer that there are no scale benefits available in its sub-technologies. Most of real-life production technologies are multi-stage in nature, and the sources of increasing returns lie in the sub-technologies. It is, therefore, imperative to estimate the scale economies of a firm not only for the network technology but also for the sub-technologies. To accomplish this, two approaches are suggested in this contribution, based on the premise concerning whether a network technology construct considers allocative inefficiency. The first approach, which is ours, makes use of a single network technology for two interdependent sub-technologies. The second approach, which is due to Kao and Hwang (2011), however, assumes complete allocative efficiency by considering two independent sub-technology frontiers, one for each sub-technology. The distinction between these two approaches is important from a policy point of view since the network efficiencies revealed from these two approaches have distinctive caustive factors that do not permit them to be used interchangeably.

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1. Introduction

Most of real-life production technologies are multi-stage in nature. Characterization of such technologies via concept like returns to scale (RTS) or scale elasticity (SE) is considered important to firm managers for the stage-specific analysis of their business decisions concerning expansion or contraction. Therefore, it is imperative to estimate the SE of a firm not only for the network technology but also for its sub-technologies so as to locate the sources of scale economies.

Data envelopment analysis (DEA), a linear programming (LP) based technique, has been widely accepted as a competent methodology to estimate the structure of production technology in both primal (production) and dual (cost) environments. See Scarf (1990) for a discussion on the analogy between economic institutions and algorithms for solving the LP problems where the simplex method is interpreted as a search for market prices that equilibrate demand for factors of production with their supply. Much of DEA literature that considers the evaluation of SE treats production technology as a black-box (see, e.g., Sahoo et al., 1999; Fukuyama, 2003; Tone and Sahoo, 2003; Banker et al., 2004; Sahoo and Tone, 2013; Zelenyuk, 2013; among others), thus completely ignoring the literature on production control problems dealing with multi-stage production technologies (see, e.g., Aburzzi, 1965; Bakshi and Arora, 1969; among others).

The DEA literature that considers modeling of multi-stage technology by linking its sub-technologies is fairly recent. To the best of our knowledge, the network structure that links sub-technologies with intermediate products in the DEA framework was first introduced by Färe (1991); was, subsequently, extended in Färe and Grosskopf (2000) and Tone and Tsutsui (2009, 2010); and was, finally, applied in Tone and Sahoo (2003), Prieto and Zofio (2007), Yu and Lin (2008), and Lewis et al. (2013), among others.

A special variant of Färe and Grosskopf’s multi-stage technology, i.e., a two-stage technology was developed in a different way by several scholars under multiplier DEA models (see, e.g., Chen and Zhu, 2004; Chen et al., 2006; Liang et al., 2006; Liang et al., 2008; Kao and Hwang, 2008; Kao, 2009; Chen et al., 2009a; Chen et al., 2009b; Chen et al., 2010; Cook et al., 2010; Kao and Hwang, 2011; Kao, 2013; Chen et al., 2013; among others). In this set up, sub-technology I consumes input resources to produce intermediate products, which are all, in turn, used as inputs to sub-technology II to produce final outputs. A further restricted variant of this two-stage structure is developed by Seiford and Zhu...
The two-stage DEA literature (Kao and Hwang, 2008; Liang et al., 2008; Kao, 2005; Chen et al., 2009a; Chen et al., 2009b; Chen et al., 2010; Kao and Hwang, 2011; Chen et al., 2013; Kao, 2013; Lewis et al., 2013; Kao and Hwang, 2014) that addresses the evaluation of the decomposition of network efficiency into the sub-technology specific efficiencies is fairly recent. This decomposition is done under the assumption of constant returns to scale (CRS). What seems to be more intriguing but has completely been overlooked is whether this decomposition can be made under the assumption of variable returns to scale (VRS). And, if the answer to this question is yes, but at a cost, then it is worth investigating what this cost amounts to, i.e., allocative inefficiency due to any sub-optimal decision by the sub-technology managers as to how much of intermediate products to produce and consume in the world of changing prices. The first contribution of this study is to address the aforementioned issue.

Another important issue related to the first one, which has also not been addressed in the two-stage DEA literature, is the decomposition of network SE into the sub-technology specific SEs. This issue is related because the SE estimation can be done only under VRS. This decomposition will help a firm manager to not only determine the scale economies of network technology but also locate their sources, which lie in the sub-technologies. To our best knowledge, Kao and Hwang (2011) are the first to propose a scheme to determine only the scale efficiency of independent sub-technologies under the two-stage setting. Therefore, the second contribution of this study is to propose a scheme to analytically show the SE of network technology as the product of those of its two sub-technologies.

For network SE estimation, two approaches may be considered based on the premise concerning whether the VRS-based network technology construct considers allocative inefficiency. In economics, the primary purpose of constructing a technology is to address allocative efficiency associated with the economic choice of how much of intermediate products to produce and consume, in addition to the economic use of primary inputs and the maximal production of final outputs. Therefore, it is necessary that not only the intermediate products are explicitly modeled, but also their optimal values are considered in the construction of sub-technologies' frontiers so that the issue of allocative efficiency, if exists, can be addressed.

Under the first approach (Approach I), which is ours, one network frontier is constructed for the two independent sub-technology frontiers, which are linked through optimal values of intermediate products. The dual pricing interpretation of the constraint that the intermediate products are freely determined in our envelopment-based network technologies is that the weights for intermediate products as inputs and outputs in our multiplier-based network technology are the same. We maintain that our multiplier-based network technology is additive.

The construct of our proposed additive network technology holds under two conditions: (1) weights for intermediate products as inputs and outputs are the same, and (2) intercept multiplier of network technology is the sum of those of the two sub-technologies. The first condition holds due to our constraint that the intermediate products are freely determined in our envelopment-based network technology. The second condition holds under the assumption that the additive network technology can inherit the properties of its sub-technologies, i.e., if the sub-technologies satisfy the properties such as no free lunch, free disposability in inputs and outputs, compactness, convexity, and returns to scale, then so does the additive network technology.

The proof of this is made in the spirit of the proof of Proposition 2.3.2 in Färe and Grosskopf (1996, p. 23, pp. 44–45). The network technical efficiency (TE) decomposition based on Approach I reveals that allocative inefficiency arises only under the VRS specification, but disappears under the CRS specification. It can, therefore, be argued that interpreting the ‘same weights’ assumption for the intermediate products as outputs and inputs as a perfect coordination between the two sub-technologies, as in Liang et al. (2008), is not sufficient to rule out allocative inefficiency in the VRS environment. Allocative inefficiency is a broader concept that includes inefficiencies arising from possible sub-optimal decisions as to how much of intermediate products to produce and consume in the world of changing prices. Our additive network technology can be used in identifying such inefficiency when optimal values of intermediate products are less than their observed values. Our network TE decomposition reveals that a network firm is fully efficient only when it is efficient in both of its sub-technologies.

The second approach (Approach II), which is due to Kao and Hwang (2011), requires the two sub-technologies to be independent for the construction of network frontier. To keep the sub-technologies independent, the input-orientation in the sub-technology I and the output-orientation in the sub-technology II are maintained to keep the level of intermediate products unaltered. This way of modeling network technology assumes the current uses of intermediate products as optimal, thereby effectively rules out allocative inefficiency arising from their possible sub-optimal uses. However, allocative inefficiency of this kind, if exists, may question the very TE estimates estimated against the two assumed independent sub-technology frontiers.

Note that the choice of a particular approach adopted implies whether assuming allocative inefficiency in the underlying technology construct, and hence, yields a distinct set of TE estimates. The distinction between the two approaches is important from a policy point of view as the factors attributing to the network’s inefficiency in each approach are distinct. For example, a lower network TE may be due to allocative inefficiency in Approach I as against the same due to lower sub-technologies’ efficiencies in Approach II. In this case, policies to remove allocative inefficiency may be more effective in improving the network efficiency in Approach I than the policies directed at improving the sub-technology specific TE.

The remainder of the paper proceeds as follows. Section 2 deals with a discussion on the development of variants of two-stage network DEA models to estimate the TE and SE of firms in the network technology as well as sub-technologies. Section 3 provides an illustrative empirical application, showing how the TE and SE estimates of a firm yielded from the two approaches are different due to allocative inefficiency. Section 4 provides some concluding remarks.

2. Model development

2.1. Two-stage network technology

Consider a two-stage technology in which sub-technologies are connected in a network to form a network technology (T^n) (see Fig. 1). Further, assume that there are n firms, and each firm (h = 1, 2, . . ., n) in the first sub-technology (T^i) uses inputs x_i (i = 1, 2, . . ., m) to produce intermediate outputs z_d (d = 1, 2, . . ., p) and the same firm in the second sub-technology (T^n) uses these intermediate outputs as inputs to produce final outputs y_r (r = 1, 2, . . ., s). These z_d are called intermediate measures by Chen and Zhu (2004) and Liang et al. (2008).
2.2. TE estimation

We now discuss the TE evaluation using Approach I.

2.2.1. TE estimation using Approach I

One can evaluate the TE of a network firm either in input-oriented manner or in output-oriented manner or in non-oriented manner. In this study we, however, concentrate on TE evaluation in input-oriented manner. We set up the following input-oriented VRS-based network DEA model for estimating the input TE of firm $h$ ($TE_{ih}\textsuperscript{(N)}$) in envelopment form as

$$ TE_{ih}\textsuperscript{(N)} = \min_{\beta, x, z} \beta_h $$

subject to

$$ \sum_{j=1}^{n} x_{ij} \beta_h j \leq \beta_h x_{ih} \forall i, $$

$$ \sum_{j=1}^{n} z_{ij} \beta_h j - \tilde{z}_ih \geq 0 \forall i, $$

$$ \sum_{j=1}^{n} \beta_h j = 1. \text{ (sub-technology I)} $$

$$ \sum_{j=1}^{n} \beta_h j \leq \tilde{y}_ih \forall r. $$

$$ \sum_{j=1}^{n} \beta_h j = 1. \text{ (sub-technology II)} $$

where $\beta_h \leq 1, \beta_h j \geq 0 \forall j, \tilde{z}_ih : \text{ free}\forall i, j$. Let $(\beta^*, \tilde{x}^*, \tilde{y}^*)$ be optimal solution vector of model (1), which is based on the following VRS-based network technology set ($T^{(VRS)}_{ih}$) defined as

$$ T^{(VRS)}_{ih} = \left\{ (x, y, z) \mid \sum_{j=1}^{n} x_{ij} \leq x_{ih} \forall i, \sum_{j=1}^{n} z_{ij} \beta_h j - \tilde{z}_ih \leq 0 \forall j, \sum_{j=1}^{n} \beta_h j = 1. \right\} $$

$T^{(VRS)}_{ih}$ uses $\beta$ and $\mu$ as intensity weights to form a linear combination of $n$ observed firms. Since both $T^{(VRS)}_{iI}$ and $T^{(VRS)}_{iII}$ satisfy VRS (i.e., $\sum_{j=1}^{n-1} \beta_h j = 1$ and $\sum_{j=1}^{n} \beta_h j = 1$), $T^{(VRS)}_{ih}$ satisfies VRS. Similarly, $T^{(VRS)}_{iI}$ and $T^{(VRS)}_{iII}$ satisfy the assumption of strong (free) disposability of inputs and outputs by the use of inequality constraints, and so is the case with $T^{(VRS)}_{ih}$. The most distinguishing feature of $T^{(VRS)}_{ih}$ is that the intermediate products are explicitly modeled to be freely determined so as to make the sub-technologies interdependent. Chen and Zhu (2004), Liang et al. (2008), and Chen et al. (2010) also used this feature to reveal the frontier points of the two-stage technology.

$\beta_h$ can be regarded as representing the minimum input proportion possible in $T^{(VRS)}_{ih}$ to produce $y_h$. Firm $h$ is technically efficient, i.e., $TE_{ih}\textsuperscript{(N)} = 1$ if and only if $(\beta_h^*, \tilde{x}_ih, \tilde{y}_ih) \in T^{(VRS)}_{ih}$ where $T^{(VRS)}_{ih}$ represents the boundary of $T^{(VRS)}_{ih}$ and $(\beta_h^*, \tilde{x}_ih, \tilde{y}_ih) \notin T^{(VRS)}_{ih}$ when $\tilde{z}_ih \neq \tilde{z}_ih$.

One can also set up the input-oriented VRS-based network DEA model for estimating the input TE of firm $h$ ($TE_{ih}\textsuperscript{(N)}$) in multiplier form as

$$ TE_{ih}\textsuperscript{(N)} = \max_{\gamma, \eta, \xi} \sum_{r=1}^{m} \gamma_ry_{ih} - \eta_h - \xi_h $$

subject to

$$ \sum_{d=1}^{p} \gamma_d z_{ih} - \sum_{i=1}^{m} \eta_i y_{ih} \leq 0 \forall i, $$

$$ \sum_{d=1}^{p} \gamma_d z_{ih} - \sum_{i=1}^{m} \eta_i y_{ih} \leq 0 \forall j. $$

where $\gamma_d, \eta_i$ and $\xi_h$ are the dual decision variables to the respective constraints of sub-technology I, and $\gamma_d z_{ih}$, $\eta_i y_{ih}$, and $\xi_h$ are the dual decision variables to the respective constraints of sub-technology II in (1). Here $\omega_d = \omega_i = \omega_h$, which is due to the constraint that $\tilde{z}_ih$ are free in (1). Otherwise, $\omega_d$ would have been less than or equal to $\omega_i$. $\omega_h$ had $\tilde{z}_ih$ been non-negative. Note that Liang et al. (2008) model the 'same weights' assumption on $\tilde{z}_ih$ as a perfect coordination between the two sub-technologies under the CRS specification.

Constraints (3.2) and (3.3) correspond to the sub-technologies $T^{(VRS)}_{iI}$ and $T^{(VRS)}_{iII}$ respectively whose respective intercept multipliers are $\eta_i$ and $\xi_h$. The construct of our network technology is such that the network technology constraint is the sum of the two sub-technologies constraints, i.e., $TE_{ih}\textsuperscript{(N)}$ is additive. This proposed additive structure holds under two conditions: (1) weights for the intermediate measures (products as inputs and outputs) are the same, and (2) intercept multiplier of $T^{(VRS)}_{ih}$ is the sum of those of its two sub-technologies. The first condition is satisfied due to the fact that $\tilde{z}_ih$ are free in (1). The second condition holds under the assumption that the additive $T^{(VRS)}_{ih}$ can inherit the properties of its sub-technologies, i.e., if the sub-technologies satisfy the properties such as no free lunch, free disposability in inputs and outputs, compactness, convexity, and returns to scale, then so does the additive network technology. The proof of this is made in the spirit of the proof of Proposition 2.3.2 in Färe and Grosskopf (1996, p. 23, pp. 44-45).

Using optimal multipliers from (3), one can obtain the input-oriented TE of firm $h$ in $T^{(VRS)}_{ih}$ ($TE_{ih}\textsuperscript{(N)}$) and the sub-technologies ($TE_{ih}\textsuperscript{(I)}$ and $TE_{ih}\textsuperscript{(II)}$) as:

$$ TE_{ih}\textsuperscript{(N)} = \frac{\sum_{d=1}^{p} \gamma_d z_{ih} - \eta_h - \xi_h}{\sum_{i=1}^{m} \eta_i y_{ih}}, \quad TE_{ih}\textsuperscript{(I)} = \frac{\sum_{d=1}^{p} \gamma_d z_{ih} - \eta_I}{\sum_{i=1}^{m} \eta_i y_{ih}} \quad \text{and} \quad TE_{ih}\textsuperscript{(II)} = \frac{\sum_{d=1}^{p} \gamma_d z_{ih} - \eta_{II}}{\sum_{i=1}^{m} \eta_i y_{ih}} $$

One can express $TE_{ih}\textsuperscript{(N)}$ as the product of three terms: the first two terms representing the TEs in the sub-technologies - $TE_{ih}\textsuperscript{(I)}$ and $TE_{ih}\textsuperscript{(II)}$, respectively, and the third term representing an index ($R_{ih}\textsuperscript{(N)}$) indicating whether the decision concerning the use of observed intermediate products (2) as intermediate measures (outputs and inputs)
inputs) is optimal, i.e., whether \( z_h \) equals \( \tilde{z}_h \). The proposed TE decomposition is given below.

\[
TE_{nh}^{(i)} = \left( \sum_{i=1}^{m} w_{ixnh} - o_{ixnh} \right) / \sum_{i=1}^{m} x_{i, nh} = \frac{\sum_{i=1}^{m} w_{ixnh} - o_{ixnh} - \sum_{i=1}^{m} x_{i, nh} (o_{ixnh} - o_{iynh})}{\sum_{i=1}^{m} x_{i, nh}} = TE_{nh}^{(i)} - \frac{1}{m} \left( \sum_{i=1}^{m} w_{ixnh} / x_{i, nh} \right)
\]

Assuming unique optimal solutions in (3), we have three remarks based on the TE decomposition in (5).

**Remark 1.** \( \lambda_{\text{VRS}} \) represents a proxy for the indication of allocative inefficiency, in which case \( \lambda_{\text{VRS}} > < 1 \). Allocative inefficiency arises under the VRS specification but disappears under the CRS specification. One can therefore infer that maintaining the ‘same weight’ assumption on \( z \) as outputs and inputs under the VRS specification is not sufficient to rule out allocative inefficiency. Allocative inefficiency is a broader concept that includes inefficiencies arising from any possible sub-optimal decision as to how much \( z \) to produce and consume in the light of changing prices, i.e., \( z_h < \tilde{z}_h \), in which case \( \lambda_{\text{VRS}} \neq 1 \). Our proposed additive \( \lambda_{\text{VRS}} \) is helpful in identifying the optimum when the optional intermediate products (\( \tilde{z}^2 \)) is less than its observed counterparts (\( z \)), i.e., \( z_h < \tilde{z}_h \) when \( \lambda_{\text{VRS}} \) turns inefficient.

**Remark 2.** \( \lambda_{\text{VRS}} = 1 \) when \( \lambda_{\text{VRS}} = 1 \), implying the decision concerning the use of observed intermediate products (\( z_h \)) as outputs and inputs as optimal, i.e., \( z_h = \tilde{z}_h \). This means that there is no allocative inefficiency in the use of observed \( z_h \). In this case, \( \lambda_{\text{VRS}} = 1 \). Therefore, the TE decomposition under the additive network structure reveals that a network firm is fully efficient only when it is efficient in both of its sub-technologies.

**Remark 3.** When \( \lambda_{\text{VRS}} < 1 \), \( \lambda_{\text{VRS}} > < 1 \). (A) \( \lambda_{\text{VRS}} > 1 \) when (1) \( o_{ixnh} < 0 \) and (2) \( \lambda_{\text{VRS}} > 1 \) when \( o_{ixnh} > 0 \) which case firm \( h \) exhibits increasing returns to scale (IRS) in \( \lambda_{\text{VRS}} \) (B) \( \lambda_{\text{VRS}} < 1 \) when \( o_{ixnh} > 0 \) in which case firm \( h \) exhibits decreasing returns to scale (DRS) in \( \lambda_{\text{VRS}} \).

To prove the statement (A) in Remark 3, let us redefine \( \lambda_{\text{VRS}} = 1 - \frac{1}{n_h} (\sum_{i=1}^{m} w_{ixnh} / x_{i, nh}) / (1 - \frac{\sum_{i=1}^{m} w_{ixnh}}{\sum_{i=1}^{m} x_{i, nh}}) \) as \( f(I, \tilde{I}) \). \( \lambda_{\text{VRS}} > 1 \) implies that \( I < \tilde{I} \). This means that \( \frac{1}{\lambda_{\text{VRS}}} < 1 \). One can see that for this strict inequality to hold, two conditions need to hold: (1) \( o_{ixnh} < 0 \) and (2) \( \lambda_{\text{VRS}} > 1 \) which case firm \( h \) exhibits IRS since \( o_{ixnh} < 0 \). Similarly, one can prove the statement (B) by examining the value of \( \lambda_{\text{VRS}} \) when it is less than 1. \( \lambda_{\text{VRS}} > 1 \) when \( I > \tilde{I} \). This inequality holds only when \( o_{ixnh} > 0 \) irrespective of the values of \( \lambda_{\text{VRS}} \), since \( \lambda_{\text{VRS}} < 1 \) and firm \( h \) exhibits DRS since \( o_{ixnh} < 0 \). Note that the issue of determination of returns to scale will be dealt with in Section 2.3.3.

Note that optimal multipliers obtained from (3) may not be unique, implying that \( TE_{nh}^{(i)} \) and \( TE_{nh}^{(ii)} \) are not unique. Therefore, in the spirit of Kao and Hwang (2008), assuming \( \lambda_{\text{VRS}} \) to be more important, we first determine the maximum value of \( \lambda_{\text{VRS}} \) via

\[
TE_{nh}^{(i)} = \text{max} \sum_{i=1}^{n} w_{ixnh} - o_{ixnh}
\]

\[
s.t. \sum_{i=1}^{n} w_{ixnh} - o_{ixnh} = \lambda_{\text{VRS}} x_{i, nh}, \sum_{i=1}^{n} x_{i, nh} = 1, \sum_{i=1}^{n} w_{ixnh} - o_{ixnh} - \sum_{i=1}^{n} x_{i, nh} (o_{ixnh} - o_{iynh}) = \sum_{i=1}^{n} w_{ixnh}/x_{i, nh} - \lambda_{\text{VRS}} \sum_{i=1}^{n} w_{ixnh}/x_{i, nh}
\]

One can then compute the minimum of \( \lambda_{\text{VRS}} \) by using optimal multipliers obtained from model (6). However, if \( \lambda_{\text{VRS}} \) is considered more important, we first determine the maximum value of \( \lambda_{\text{VRS}} \) and then the minimum value of \( \lambda_{\text{VRS}} \) in an analogous manner.

We now illustrate how to measure TE using Approach II.

2.2.2. TE estimation using Approach II

As shown in Chen et al. (2013), since the sub-technology specific TEs can be computed independently of the overall efficiencies, we set up the network technology set \( \{\lambda_{\text{VRS}}\} \):

\[
\quad \lambda_{\text{VRS}} = \{x, y, \lambda_{\text{VRS}} \} \cup \{\lambda_{\text{VRS}} \}
\]

where

\[
\lambda_{\text{VRS}} = \{x, y, \sum_{i=1}^{n} w_{ixnh} - o_{ixnh} - \sum_{i=1}^{n} x_{i, nh} (o_{ixnh} - o_{iynh}) = \sum_{i=1}^{n} w_{ixnh}/x_{i, nh} - \lambda_{\text{VRS}} \sum_{i=1}^{n} w_{ixnh}/x_{i, nh}
\]

For the construction of \( \lambda_{\text{VRS}} \), Kao and Hwang (2011) maintains input-orientation in \( \lambda_{\text{VRS}} \) and output-orientation in \( \lambda_{\text{VRS}} \). The input-oriented TE of firm \( h \) in \( \lambda_{\text{VRS}} \) can be computed by setting up the following linear program:

\[
\quad \lambda_{\text{VRS}} = \min_{\lambda_{\text{VRS}}} \{\delta_h : (x, y, \lambda_{\text{VRS}}) \in \lambda_{\text{VRS}} \}
\]

Similarly, the output-oriented TE of firm \( h \) in \( \lambda_{\text{VRS}} \) can be obtained from the following linear problem:

\[
\quad \lambda_{\text{VRS}} = \max_{\lambda_{\text{VRS}}} \{\mu_h : (x, y, \lambda_{\text{VRS}}) \in \lambda_{\text{VRS}} \}
\]

Kao and Hwang (2011) have shown that the network TE of firm \( h \) \( \lambda_{\text{VRS}} \) is the product of \( \lambda_{\text{VRS}} \) and \( \lambda_{\text{VRS}} \), i.e.,

\[
\lambda_{\text{VRS}} = \lambda_{\text{VRS}} \times \lambda_{\text{VRS}}
\]

We now discuss the evaluation of SE.

2.3. SE evaluation

2.3.1. Estimating SE using Approach I

To compute the input-oriented SE of network firm \( h \), we first need to compute its TE using the model (1). Let its optimal solution vector be \( \{\hat{w}_h, \hat{z}_h, \hat{y}_h \} \). Firm \( h \) is (input-oriented) technically efficient if \( \hat{y}_h = 1 \), \( \hat{z}_h = \tilde{z}_h \) and input and output slacks are all zero. If it is not, then it needs to be projected onto the network frontier by applying the following formulae:

\[
x_h' = \hat{y}_h \hat{x}_h - z', \quad \tilde{z}_h = \tilde{y}_h - s', \quad y_h' = y_h + s'
\]

where \( s' \) and \( s' \) are respective vectors of input and output slacks under (1).
Due to duality theory, the following transformation function
\[ P^{(0)}(x_h, y_h; z_h) = 0 \]
holds:
\[ P^{(0)}(x_h, y_h; z_h) \equiv - \sum_{i=1}^{m} \lambda_{i}^{*}(y_{ih} + s_{i}^{*}) - \sum_{i=1}^{m} \lambda_{i}^{*}(\beta_{i}^{*}x_{ih} - s_{i}^{*}) - \alpha_{h}^{*} - \omega_{h}^{*} = 0 \]
(12)
where \(\lambda_{i}^{*}, \lambda_{i}^{*}, \omega_{i}^{*}\) and \(\omega_{i}^{*}\) are assumed to be the unique optimal multipliers obtained from (3); otherwise \(P^{(0)}(\cdot)\) is not differentiable at extreme points.

To define the SE in \(T_{\text{VRS}}^{(0)}\), and \(T_{\text{VRS}}^{(0)}\), we consider, respectively, the following input–output vectors from (11): \((x_{h}, y_{h}), (x_{h}, z_{h})\) and \((z_{h}, y_{h})\). Following Baumol et al. (1982), we define the input-oriented (local) SE of firm \(h\) in \(T_{\text{VRS}}^{(0)}\) as:
\[ e_{h}^{(0)}(x_{h}, y_{h}; z_{h}) \equiv - \sum_{i=1}^{m} \frac{\partial P^{(0)}(\cdot)}{\partial x_{ih}} \bigg/ \sum_{i=1}^{m} \lambda_{i}^{*} = \frac{\beta_{h}^{*}}{\beta_{h}^{*} + \alpha_{h}^{*} + \omega_{h}^{*}} \]
(13)
Note that in (13), \(\sum_{i=1}^{m} \lambda_{i}^{*} = 1\) due to (3.1); and \(\sum_{i=1}^{m} \lambda_{i}^{*} = \beta_{h}^{*} + \alpha_{h}^{*} + \omega_{h}^{*}\), due to duality, the objective function values of (1) and (3) are the same, i.e., \(\beta_{h}^{*} = \sum_{i=1}^{m} \lambda_{i}^{*} y_{ih} - \alpha_{h}^{*} - \omega_{h}^{*}\).
Based on (13), we have now the following proposition.

Proposition 1. The input-oriented network returns to scale are increasing (IRS) (i.e., \(e_{h}^{(0)} > 0\)) if \(\alpha_{h}^{*} + \omega_{h}^{*} < 0\) in all optimal solutions, constant (CRS) (i.e., \(e_{h}^{(0)} = 0\)) if \(\alpha_{h}^{*} + \omega_{h}^{*} = 0\) in an optimal solution, and decreasing (DRS) (i.e., \(e_{h}^{(0)} < 1\)) if \(\alpha_{h}^{*} + \omega_{h}^{*} > 0\) in all optimal solutions.

Proof 1. The proof is similar to that of determining the RTS underlying black-box DEA model. See Banker and Thrall (1992) and Banker et al. (2004).

We now discuss the analytical SE evaluation of a fully network efficient firm \(h\) in its sub-technologies for which the constraints (3.2) and (3.3) are of special interest. Note that the network technology constraint is the sum of its two sub-technology constraints (3.2) and (3.3), i.e.,
\[ \sum_{i=1}^{m} \lambda_{i}^{*}(y_{ih} + s_{i}^{*}) - \sum_{i=1}^{m} \lambda_{i}^{*}(\beta_{i}^{*}x_{ih} - s_{i}^{*}) - \alpha_{h}^{*} - \omega_{h}^{*} = \]
\[ = \left( \sum_{i=1}^{m} \lambda_{i}^{*}y_{ih} + s_{i}^{*} - \sum_{i=1}^{m} \lambda_{i}^{*}\beta_{i}^{*}x_{ih} - s_{i}^{*}\right) - \alpha_{h}^{*} - \omega_{h}^{*} \]
\[ + \left( \sum_{i=1}^{m} \lambda_{i}^{*}x_{ih} - s_{i}^{*} - \sum_{i=1}^{m} \lambda_{i}^{*}x_{ih} = \right) \]
(14)
Since \(\sum_{i=1}^{m} \lambda_{i}^{*}y_{ih} - \sum_{i=1}^{m} \lambda_{i}^{*}x_{ih} - s_{i}^{*} - \omega_{h}^{*} = 0\) for the technically efficient firm \(h\) in \(T_{\text{VRS}}^{(0)}\), \(h\) will also be efficient in \(T_{\text{VRS}}^{(0)}\) and \(T_{\text{VRS}}^{(0)}\), in which case the respective transformation functions are:
\[ P^{(0)}(x_{h}, y_{h}) \equiv - \sum_{i=1}^{m} \lambda_{i}^{*}(y_{ih} + s_{i}^{*}) - \sum_{i=1}^{m} \lambda_{i}^{*}(\beta_{i}^{*}x_{ih} - s_{i}^{*}) - \alpha_{h}^{*} = 0 \]
(15)
\[ P^{(0)}(x_{h}, y_{h}) \equiv - \sum_{i=1}^{m} \lambda_{i}^{*}(y_{ih} + s_{i}^{*}) - \sum_{i=1}^{m} \lambda_{i}^{*}(\beta_{i}^{*}x_{ih} - s_{i}^{*}) - \alpha_{h}^{*} = 0 \]
(16)
Using (13), one can obtain the respective sub-technology specific input-oriented SEs as:
\[ e_{h}^{(0)}(x_{h}, y_{h}) \equiv - \sum_{i=1}^{m} \lambda_{i}^{*}(y_{ih} + s_{i}^{*}) - \sum_{i=1}^{m} \lambda_{i}^{*}(\beta_{i}^{*}x_{ih} - s_{i}^{*}) - \alpha_{h}^{*} = 0 \]
(17)
Similarly, one can use the SE expression (18) to determine the left- and right-hand SEs of firm \(h\) in \(T_{\text{VRS}}^{(0)}\) as:
\[ e_{h}^{(0)}(x_{h}, y_{h}) \equiv \frac{\beta_{h}^{*}}{\beta_{h}^{*} + \alpha_{h}^{*}} \quad \text{and} \quad e_{h}^{(0)}(x_{h}) = \frac{\beta_{h}^{*}}{\beta_{h}^{*} + \alpha_{h}^{*} + \omega_{h}^{*}} \]
(21)
Similarly, one can use the SE expression (18) to determine the left- and right-hand SEs of firm \(h\) in \(T_{\text{VRS}}^{(0)}\) as:
\[ e_{h}^{(0)}(x_{h}, y_{h}) \equiv \frac{\beta_{h}^{*} + \alpha_{h}^{*}}{\beta_{h}^{*} + (\alpha_{h}^{*} + \omega_{h}^{*})} \quad \text{and} \quad e_{h}^{(0)}(x_{h}) = \frac{\beta_{h}^{*} + \omega_{h}^{*}}{\beta_{h}^{*} + (\alpha_{h}^{*} + \omega_{h}^{*})} \]
(22)
While defining these sub-technology specific SEs, we have followed Banker and Thrall (1992) to consider the upper and lower bounds of \((\alpha_{h}^{*} + \omega_{h}^{*})\) in the program (20), i.e., \((\alpha_{h}^{*}, 0)\) and \((\alpha_{h}^{*}, \omega_{h}^{*})\) to determine the left- and right-hand SEs. However, if one considers the individual max (min) values of \(\alpha_{h}^{*}\) and \(\omega_{h}^{*}\) (i.e., \(\alpha_{h}^{*}(\alpha_{h}^{*})\) and \(\omega_{h}^{*}(\omega_{h}^{*})\), which can be obtained by replacing max (min) \((\alpha_{h}^{*} + \omega_{h}^{*})\)
in the objective of (20) with max (min)\(x_I\) and max (min)\(x_{II}\) respectively, then our SE expressions in (21) and (22) may produce the incorrect values of left- and right-hand SEs. This is possible only when \(\delta_h(x_{II}) \neq \alpha_{II}(\alpha_{II})\) and \(\delta_h(x_{II}) \neq \alpha_{II}(\alpha_{II})\).

We have now our Proposition 2.

**Proposition 2.**

(2.1) Assuming alternate optima in \(x_I + x_{II}\), \(T_{III}^{(II)}(\cdot)\) exhibits IRS 
\[ (\rho_{III}^{(II)}(\cdot) > 1) \quad \text{if} \quad (\alpha_{II} + \omega_{II}) < 0 \quad \text{CRS} \quad (\rho_{III}^{(II)}(\cdot) < 1) \quad \text{if} \quad (\alpha_{II} + \omega_{II}) > 0 \quad \text{and} \quad \text{DRS} \quad (\rho_{III}^{(II)}(\cdot) < 1) \quad \text{if} \quad (\alpha_{II} + \omega_{II}) > 0. \]

(2.2) Assuming alternate optima in \(x_I + x_{II}\), \(T_{III}^{(II)}(\cdot)\) exhibits IRS 
\[ (\rho_{III}^{(II)}(\cdot) > 1) \quad \text{if} \quad \omega_I > 0, \quad \text{CRS} \quad (\rho_{III}^{(II)}(\cdot) < 1) \quad \text{if} \quad \omega_I < 0, \quad \text{and} \quad \text{DRS} \quad (\rho_{III}^{(II)}(\cdot) < 1) \quad \text{if} \quad \omega_I > 0. \]

(2.3) Assuming alternate optima in \(x_I + x_{II}\), \(T_{III}^{(II)}(\cdot)\) exhibits IRS 
\[ (\rho_{III}^{(II)}(\cdot) > 1) \quad \text{if} \quad \omega_{II} < 0, \quad \text{CRS} \quad (\rho_{III}^{(II)}(\cdot) < 1) \quad \text{if} \quad \omega_{II} > 0, \quad \text{and} \quad \text{DRS} \quad (\rho_{III}^{(II)}(\cdot) < 1) \quad \text{if} \quad \omega_{II} > 0. \]

**Proof.** The proof is similar to that of determining the RTS underlying block-black DEA model. See Banker and Thrall (1992) and Banker et al. (2004).

Banker et al. (1984) are the first to show that the intercept \(c_I\) in the multiplier form of the black-box DEA model can be used to estimate RTS. Several contributions exist, at the extreme points, on estimating the right-hand (upper bound) and left-hand SE (lower bound) measures in the black-box models. See, e.g., among others, Banker and Thrall (1992), Farsund (1996), Tone and Sahoo (2004), Tone and Sahoo (2005), Tone and Sahoo (2006), Hadjicostas and Soteriou (2006), Podinovski et al. (2009), and Sahoo et al. (2012).

2.3.2. Estimating SE using Approach II

To compute the input-oriented SE of firm \(h\) in \(T_{III}^{(III)}\), we first set up the dual of model (8) as

\[ T_{III}^{(II)}(\cdot) = \max \sum_{d=1}^{m} w_d x_d - \alpha_{II} \tag{23} \]

s.t.
\[ \sum_{i=1}^{n} \nu_i x_i = 1, \tag{23.1} \]
\[ \sum_{d=1}^{m} w_d x_d - \sum_{i=1}^{n} \nu_i x_i - \alpha_{II} \leq 0 \quad (\forall j), \tag{23.2} \]
\[ \nu, w_d \geq 0 \quad (\forall i, d); \alpha_{II} : \text{free}. \tag{23.3} \]

Assume that unique optimal solutions in (23) exist. The duality theory suggests that the following transformation function for firm \(h\), \(P_{II}(z_{III})\) in (8) holds, i.e.,

\[ \begin{align*} P_{II}(z_{III}) &= \sum_{d=1}^{m} w_d x_d + \sum_{i=1}^{n} \nu_i (\delta_{II} x_i - s_I) - \omega_{II} = 0 \tag{24} \end{align*} \]

where \(s_I\) and \(s_{II}\) are respectively the \(d_I\) input and \(d_{II}\) output slack variables in model (8). Using the SE formula (13), one can obtain the input-oriented SE of firm \(h\) in \(T_{III}^{(III)}\) as

\[ \begin{align*} \delta_{II}^{(II)}(z_{III}) &= \sum_{i=1}^{n} \frac{\partial P_{II}(\cdot)}{\partial x_i} / \sum_{d=1}^{m} w_d x_d = \delta_{II} + \alpha_{II} \tag{25} \end{align*} \]

Notice that in (25), \(\sum_{i=1}^{n} \nu_i x_i = 1\) due to (23.1); and \(\sum_{d=1}^{m} w_d z_{II} = \delta_{II} + \omega_{II}\), which is because, by duality, the objective function values of (8) and (23) are the same, i.e.,

\[ \delta_{II}^{(II)} = \sum_{d=1}^{m} w_d z_{II} - \omega_{II}. \]

One can compute the output-oriented SE of firm \(h\) in \(T_{III}^{(III)}\) by setting up the dual of (9) as

\[ \begin{align*} T_{III}^{(II)}(\cdot) &= \min \sum_{d=1}^{m} w_d x_d + \omega_{II} \tag{26} \end{align*} \]

s.t.
\[ \sum_{i=1}^{n} \nu_i x_i = 1, \tag{26.1} \]
\[ \sum_{d=1}^{m} w_d x_d - \sum_{i=1}^{n} \nu_i y_i + \omega_{II} \geq 0 \quad (\forall j), \tag{26.2} \]
\[ \nu, w_d \geq 0 \quad (\forall i, d); \omega_{II} : \text{free} \tag{26.3} \]

Assume that unique optimal solutions in the model (26) exist. Due to duality theory, the following transformation function for firm \(h\), \(P_{II}(z_{III})\) in (8) holds, i.e.,

\[ \begin{align*} P_{II}(z_{III}) &= \sum_{i=1}^{n} \nu_i (\delta_{II} x_i + s_I) - \sum_{d=1}^{m} w_d (z_{II} - s_{II} - \omega_{II}) = 0 \tag{27} \end{align*} \]

where \(s_I\) and \(s_{II}\) are respectively the \(d_I\) input slack and \(d_{II}\) output slacks of the model (9). Using the SE formula (13), one can obtain the output-oriented SE of firm \(h\) in \(T_{III}^{(III)}\) as

\[ \begin{align*} \delta_{II}^{(II)}(z_{III}) &= \sum_{d=1}^{m} w_d z_{II} - \mu_{II} \sum_{i=1}^{n} \nu_i y_i = \mu_{II} - \delta_{II} - \omega_{II} \tag{28} \end{align*} \]

Notice that in (28), \(\sum_{i=1}^{n} \nu_i y_i = 1\) due to (26.1); and \(\sum_{d=1}^{m} w_d z_{II} = \mu_{II} - \delta_{II}\), which is because, by duality, the objective function values of (9) and (26) are the same, i.e.,

\[ \mu_{II} = \sum_{d=1}^{m} w_d z_{II} + \omega_{II}. \]

Since in many cases \(\delta_{II}\) and \(\omega_{II}\) are not uniquely determined in (23) and (26) respectively, the SE estimates are not unique. There is thus a need to find out both right- and left-hand SEs.

We set up the following model to compute the input-oriented right-hand SE of firm \(h\), \(\delta_{II}^{(II)}(z_{III})\) in (8) as

\[ \begin{align*} \delta_{II}^{(II)}(z_{III}) &= \max \sum_{i=1}^{n} \nu_i (\delta_{II} x_i - s_I) - \omega_{II} \tag{29} \end{align*} \]

Denote optimal solution of \(\delta_{II}^{(II)}(z_{III})\) as \(\delta_{II}^{(II)}\). \(\delta_{II}^{(II)}(\cdot)\) can be computed as:

\[ \delta_{II}^{(II)} = \frac{\delta_{II}}{\delta_{II} + \omega_{II}} \tag{30} \]

Similarly, the input-oriented left-hand SE of firm \(h\), \(\delta_{II}^{(II)}(z_{III})\) in (8) can be computed by running the model (29) with ‘min’ instead of ‘max’.

The output-oriented right-hand SE of firm \(h\) in \(T_{III}^{(III)}\) can be computed by setting up the following linear problem:
\[ \mu_h \left( 1 - \frac{\mu_h^{(i)}}{\mu_h} \right) = \max \omega_h \]

\[
\text{s.t.} \quad \sum_{r=1}^{s} u_r \left( \mu_h y_{rh} + s_r^{*} \right) = 1, \\
\sum_{d=1}^{s} w_d \left( z_{dh} - s_d \right) - \sum_{r=1}^{s} u_r \left( \mu_h y_{rh} + s_r^{*} \right) + \omega_h = 0, \\
\sum_{d=1}^{s} w_d z_{dh} - \sum_{r=1}^{s} u_r y_{rj} + \omega_h \geq 0 \quad (\forall j \neq h), \\
u_r, w_d \geq 0 \quad \forall r, d \quad \text{and} \quad \omega_h: \text{free}. \quad (31)\]

Denote optimal solution of \( \omega_h \) in (31) as \( \omega_h^{*} \). The output-oriented right-hand SE of firm \( h \) in \( T^{(0)}_{\text{WR}} \) can be computed as

\[ \nu_h^{(o)(j)}(\cdot) = 1 - \frac{\alpha_h^{(o)}}{\mu_h} \quad (32) \]

Similarly, the output-oriented left-hand SE of firm \( h \), \( \nu_h^{(l)(j)}(\cdot) \) in \( T^{(0)}_{\text{WR}} \) can be computed by running (31) with ‘min’ instead of ‘max’.

Note that unlike in Approach I, it is not possible in Approach II to decompose the network technology SE into its sub-technology specific SEs. We, however, note that Kao and Hwang (2011) develop an ad hoc approach to obtain network scale efficiency as the product of the sub-technology specific scale efficiencies.

### 3. An illustrative example

Consider a simple hypothetical data set exhibited in Table 1. There are nine firms labeled as A, B, C, D, E, F, G, H and I. Each firm in \( T^* \) uses one input (\( x \)) to produce an intermediate product (measure) (\( z \)), which is then taken as input to \( T^I \) by the same firm to produce one final output (\( y \)).

<table>
<thead>
<tr>
<th>Firms</th>
<th>( x )</th>
<th>( z )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>D</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>F</td>
<td>4</td>
<td>3.5</td>
<td>2</td>
</tr>
<tr>
<td>G</td>
<td>5.5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>H</td>
<td>4.5</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>I</td>
<td>7</td>
<td>6.5</td>
<td>6.5</td>
</tr>
</tbody>
</table>

**Table 1** Example data set.

### 3.1. On TE estimates

Based on Table 1, Fig. 2 exhibits the two independent sub-technology frontiers in a counterclockwise orientation under the VRS specification. These frontiers are drawn by keeping \( z \) unaltered. Fig. 3 exhibits the BB frontier involving observed \( x \) and \( y \) under an appropriate RTS specification (identified with lines: A–D–H–C), and the network production frontiers revealed from both approaches (model (1) under Approach I and models (8) and (9) under Approach II).

Since the BB technology considers only the relation between inputs and final outputs, and makes no assumptions regarding the internal operations of firm, it provides no insights regarding the locations of inefficiency and scale economies. For example, firms – D and H that appear efficient in the BB technology turn out to be inefficient in the network technologies (identified with broken lines: A–F–D–B–I–G–E–C under Approach I and A–B–D–F–E–G–B–I–C under Approach II). Note that superscripts – 1 and 2 indicate, respectively, the projected points of the corresponding inefficient firms in both approaches. Points such as \( E^1 \), \( E^2 \) and \( H^1 \) are the same projected point for firm \( E \) (under both approaches) and for \( H \) (under Approach I). Similarly, points – \( B^2 \) and \( H^2 \) are the same projected point for firms – B and H under Approach II. As regards the RTS, D that appears exhibiting CRS in the BB technology exhibits IRS (if projected in an input-oriented manner) in the network technologies.

We report in Table 2 the TE decomposition results obtained from Approach I (top part) and Approach II (bottom part), which will facilitate managerial insights regarding specific area of improvement for the network inefficient firms. The upshot of these results is summarized below.

1. Both approaches are in complete agreement in identifying the network efficient firms. The examples of such firms are A, C and E.
2. As expected, \( \mu_h^{(o)} \) is greater than 1 for those firms (B, D, F, G, H and I) that are technically inefficient in \( T^{(0)}_{\text{WR}} \). Technical inefficiency arises only when the intermediate products consumed by these firms are not minimal implying that there is an overproduction of these outputs in \( T^{(0)}_{\text{WR}} \). The results of our model (1) reveal that the optimal quantities of these products (\( z^* \)) are 2.8 (6), 2.2 (2.5), 1.6 (3.5), 3.4 (5), 4 (6) and 5.5 (6.5) for B, D, F, G, H and I respectively (the terms in brackets are their respective actual quantities). This is why the estimated sub-technology frontiers in Figs. 4 and 5 are different from those in Fig. 2.
finds five firms – A, B, D, F and G operating under IRS, two firms – E and H under CRS. While the sources of increasing returns of firms in $T_{VRS}^{N(I)}$ are all located in both of the sub-technologies, the same is not the case for firms exhibiting decreasing and/or constant returns. For example, $T_{VRS}^{N(I)}$ exhibiting DRS for firm I is due to DRS in $T_{VRS}^{N(I)}$ even though CRS prevails in $T_{VRS}^{N(II)}$. Similarly, $T_{VRS}^{N(II)}$ exhibiting CRS for firms – E and H is precisely due to CRS in $T_{VRS}^{N(I)}$ even though IRS prevails in $T_{VRS}^{N(II)}$.

3.2.1. SE estimates using Approach I

Using Approach I we run both max and min forms of the model (29), which is based on the optimal input TE values of the model (8), to compute the input-oriented left- and right-hand SEs of the firms in $T_{VRS}^{N(I)}$. Similarly, we run the both max and min forms of the model (31), which is based on the optimal output TE values of the model (9), to compute the output-oriented left- and right-hand SEs of the firms in $T_{VRS}^{N(II)}$. However, under this approach, it is not possible to compute the input-oriented network SEs of firms using (29) and (31). Therefore, in order to compute the input-oriented left- and right-hand network SEs, we use firms’ projected input–output vectors, $(x, y)$, obtained from (8) and (9), in model (29). The input-oriented network SE estimates are reported in Table 3 (bottom part). We find five firms – A, B, D, F and G operating under IRS, two firms – E and H under CRS and the remaining two firms – C and I under DRS (which all can be visualized in Fig. 3).

Note that since it is not possible in this approach to decompose network SE into its sub-technology specific SEs, the scale economies/diseconomies revealed from sub-technologies [(29) and (31)] may not attribute to the network scale economy/diseconomy obtained from the use of projected data of network firms $(x, y)$ in (29). For example, consider firm B whose sub-technologies exhibit CRS and DRS (CRS in $T_{VRS}^{N(I)}$ and DRS in $T_{VRS}^{N(II)}$), but its network technology, $T_{VRS}^{N(II)}$ exhibits IRS. It is therefore quite improbable to argue that the sources of increasing returns in the network technology are due to CRS and DRS in the sub-technologies. Note that the very purpose of computing the input-oriented network SE of firms under Approach II is just to compare these SE estimates with those obtained under Approach I.

Notice that though the network technologies revealed from the both approaches look similar (see Fig. 3), and the (input-oriented) RTS possibilities of network firms are the same; the degrees of underlying SE estimates of some network firms are different due to the differential nature (flatness/steepness) of some production facets. For example, $T_{VRS}^{N(I)}$ finds B exhibiting IRS whose value ranges...

---

Table 2

<table>
<thead>
<tr>
<th>Firms</th>
<th>$TE_{VRS}^{N(I)}$</th>
<th>$TE_{VRS}^{N(II)}$</th>
<th>$IVRS_{k}$</th>
</tr>
</thead>
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<tr>
<td><strong>Approach I</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
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</tr>
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<td>C</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
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<td>1</td>
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</tr>
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<td>G</td>
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</tr>
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<td>H</td>
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</tr>
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</tr>
<tr>
<td><strong>Approach II</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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</tr>
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<td>B</td>
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</tr>
<tr>
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<td>1</td>
</tr>
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<td>D</td>
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<tr>
<td>I</td>
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<td>0.714</td>
<td>0.951</td>
</tr>
</tbody>
</table>

3. The finding that the two approaches yield differential TE decomposition results for network inefficient firms is not at all strange. As expected, the decision to allow allocative inefficiency into the system in Approach I yields a frontier different from the one yielded from Approach II with no allocative inefficiency.

We now discuss in the immediately following section the sources of input-oriented scale effects of network firms within the sub-technologies.

3.2. On SE estimates

3.2.2. SE estimates using Approach II

Using Approach II we run both max and min forms of the model (29), which is based on the optimal input TE values of the model (8), to compute the input-oriented left- and right-hand SEs of the firms in $T_{VRS}^{N(I)}$. Similarly, we run the both max and min forms of the model (31), which is based on the optimal output TE values of the model (9), to compute the output-oriented left- and right-hand SEs of the firms in $T_{VRS}^{N(II)}$. The SE results are reported in Table 3 (top part). The results reveal that $T_{VRS}^{N(I)}$ finds five firms – A, B, D, F and G operating under IRS, two firms – E and H under CRS and two firms – C and I under DRS. While the sources of
from 1.583 to 2 since its SE is estimated against the vertex point B\textsuperscript{1} connecting two facets – D\textsuperscript{1}B\textsuperscript{1} and B\textsuperscript{1}G\textsuperscript{1}. \(T_{(II)}^{VRS}\) also finds this firm operating under the same IRS but its SE value is now exact at 1.583 since it is estimated against a point on the facet D\textsuperscript{2}F\textsuperscript{2}. So are the cases with firms – D and I.

On comparison between the two approaches with regard to the sources of scale economies of firms, we find some divergent information on their RTS possibilities. Though both approaches maintain input-orientation in \(T_{(I)}^{VRS}\), they yield contrasting RTS possibilities for some firms. For example, while \(T_{(I)}^{VRS}\) finds both B and H operating under IRS, and I under CRS, \(T_{(II)}^{VRS}\) finds B and H under CRS, and I under DRS. These contrasting RTS information are because the estimated \(T_{(I)}^{VRS}\) revealed from the both approaches are different (see Fig. 2 (right) and Fig. 4). However, there are contrasting information on the RTS possibilities in \(T_{(II)}^{VRS}\) even though the estimated sub-technology II frontiers are exactly the same in the both approaches (see Fig. 2 (left) and Fig. 5). This is simply due to the different orientations maintained in \(T_{(I)}^{VRS}\) for the measurement of efficiency and scale elasticity (i.e., the input orientation in \(T_{(I)}^{VRS}\) and the output orientation in \(T_{(II)}^{VRS}\)). Note that the finding that the estimated sub-technology frontiers in \(T_{(II)}^{VRS}\) are the same in the both approaches is just a coincidence.

Finally, the finding that firms – E and H exhibit CRS in the network technology and IRS and CRS in the sub-technology I and sub-technology II respectively reminds one that the CRS assumption maintained in the neoclassical theory for justifying the black-box structure of production technology does not necessarily allow one to infer that there are no scale benefits available in the sub-technologies. One can, therefore, argue that it is crucial for the firm’s ownership to locate the sources of scale effects in their sub-technologies, which will enable the firm management to improve productivity.

However, the modeling of a firm technology by considering only the inputs consumed and the final outputs produced often yields the imprecise estimates of production function; and as a result, yields erroneous inferences concerning the RTS behavior of firms (see, e.g., D and H in Fig. 3). This is because the black-box characterization obscures important relations by ignoring the interdependencies that exist between the sub-technologies.
4. Concluding remarks

In order to locate the sources of efficiency and scale economies of a network firm, two approaches are suggested, based on the premise as to whether the two-stage network technology structure considered in each approach allows allocative inefficiency. The first approach is developed by making use of a single network technology for the two interdependent sub-technologies. This approach allows for allocative inefficiency, which may arise due to any sub-optimal decision as to how much of intermediate products to produce and consume by the sub-technology managers in the world of changing prices. In the second approach, however, the technology structure is determined by assuming its sub-technologies to be independent, implying that there is no allocative inefficiency.

The current study points to future theoretical research in two directions. The first theoretical extension is the one where one could interpret a two-stage network firm as a multi-product firm producing both intermediate products and final outputs, and then, measure economies of scope by linking it with $VRS_I$, a proxy for the indication of allocative inefficiency. Instances of real-life firms suffering from profit loss due to allocative inefficiency are not usually uncommon. Therefore, production managers are given incentives to choose right output-and input-mixes so as to improve upon profit in the world of changing prices. Even if managers are not held responsible for the changing prices, management would still like to know the opportunity cost of using the sub-optimal input-and output mixes so as to improve upon profit in the world of changing prices. Even if managers are not held responsible for the changing prices, management would still like to know the opportunity cost of using the sub-optimal input-and output mixes so as to improve upon profit in the world of changing prices.

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References


Table 3

Upper and lower bounds of SE estimates.

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Approach II

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