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Short Communication

A note on two-stage network DEA model: Frontier projection and duality

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ABSTRACT

In Chen, Cook, Kao, and Zhu (2013), it is demonstrated, as a network DEA pitfall, that while the multiplier and envelopment DEA models are dual models and equivalent under the standard DEA, such is not necessarily true for the two types of network DEA models in deriving divisional efficiency scores and frontier projections. As a reaction to this work, we demonstrate that the duality in the standard DEA naturally migrates to the two-stage network DEA. Formulas are developed to obtain frontier projections and divisional efficiency scores using a DEA model's and its dual solutions. The case of Taiwanese non-life insurance companies is revisited using the newly developed approach.

1. Introduction

Data envelopment analysis (DEA) is an approach for measuring relative efficiency or calculating a composite benchmarking index when multiple performance measures (or inputs and outputs) are present in decision making units (DMUs). In recent years, a significant research has been done on DMUs with internal structures. See Castelli et al. (2010), Cook, Liang, and Zhu (2010), and Kao (2014) for excellent reviews on this field. Among a wide variety of internal structures studied, one basic and popular internal structure is called a (basic) two-stage network process where outputs from the first stage (referred to as *intermediate measures*) become the inputs to the second stage.

While several approaches have been suggested to assess the efficiency of two-stage network processes in the literature, one simple approach (referred to as the standard DEA approach) is to deal with two individual stages as independent DMUs and then measure their efficiencies separately. For example, Seiford and Zhu (1999) evaluate the performance of US commercial banks under an independent two-stage process structure, where the first stage referred to as profitability uses labor and assets to produce profits and revenue, and subsequently the second stage referred to as marketability transforms the profits and revenue into market value, returns and earnings per share. Sexton and Lewis (2003) evaluate the performance of 30 teams in two Major League Baseball leagues, whose operations are seen as a two-stage process of the front-office and on-field operations. Chilingerian and Sherman (2004) examine the performance of physician care by considering it as a two-stage process; the first stage is the manager-controlled production where the hospital managers set up and manage the assets of the hospitals, and the second stage is the physician-controlled production where the physicians decide how and when to utilize these assets to provide the medical service to the patients. All of these studies apply the standard DEA models, such as CCR (Charnes, Cooper, & Rhodes, 1978) or BCC (Banker, Charnes, & Cooper, 1984) models, to measure the individual stages' and system's efficiency scores, separately and independently.

Although the standard DEA approach discussed above are simple and convenient to use, they may give rise to potential conflicts between the two stages arising from the intermediate measures. For instance, while the second stage may need to reduce its inputs (i.e., intermediate measures) in order to attain efficiency, such an adjustment would imply a reduction in the first stage outputs, thereby deteriorating that stage's efficiency (Cook et al., 2010). To address this conflict issue, various DEA models have been suggested including notably Kao and Hwang (2008), which develop what is called centralized model in Liang, Cook, and Zhu (2008). The key features of their model are that the overall DMU efficiency is decomposed into a product of two stages' efficiency scores, and that the intermediate measures are given the same weights no matter whether they are considered inputs or outputs. Liang et al. (2008) also develop two leader-follower models when either the first or second stage is assumed to be the "leader" in performance evaluation. Kao and Hwang (2008) claim that their model is more reliable in measuring the efficiencies and consequently is capable of identifying the causes of

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inefficiency more accurately than the standard DEA approach, but they do not provide a way to obtain frontier projections for inefficient DMUs, which are required for performance benchmarking.

Chen, Cook, and Zhu (2010) are the first to note that the approach of Kao and Hwang (2008) or the centralized model of Liang et al. (2008) does not produce frontier projections for inefficient DMUs using the efficiency scores. They propose to solve an additional envelopment type of DEA model that is equivalent (or dual) to the centralized model to determine optimal values for the intermediate measures. Furthermore, in Chen, Cook, Kao, and Zhu (2013), it is demonstrated, as a network DEA pitfall, that while the multiplier and envelopment DEA models are dual models and equivalent under the standard DEA, such is not necessarily true for the two types of network DEA models when deriving information for stage or divisional efficiency scores and frontier projections.

As a reaction to these works, the current study demonstrates that the duality in the standard DEA naturally migrates to the two-stage network DEA. We develop formulas that use the linear program dual variables to calculate system and stage efficiency scores and frontier projections. As such, our proposed approach improves and simplifies the procedure outlined in Chen et al. (2010). The rest of the paper unfolds as follows. Section 2 presents a generic two-stage network system, Kao and Hwang's model for measuring its efficiency, and Chen et al.'s (2010) procedure for deriving the DEA frontier. Section 3 is devoted to the development of simple formulas for frontier projections for inefficient DMUs, followed by a discussion of the relationship between frontier projections and efficiency scores under the two-stage network structure in Section 4. Section 5, as a numerical illustration, revisits the performance evaluation of the Taiwanese non-life insurance companies studied in Kao and Hwang (2008). Section 6 concludes.

2. Two-stage network system and efficiency measurement

Consider the generic two-stage process as shown in Fig. 1, for each of a set of n DMUs. A conventional description of the process in the literature is as follows: Each DMU j ($j = 1, 2, \ldots, n$) has m inputs x_{ij} ($i = 1, 2, \ldots, m$) to the first stage, and D outputs z_{dj}, ($d = 1, 2, \ldots, D$) from that stage. These D outputs then become the inputs to the second stage and will be referred to as intermediate measures. The outputs from the second stage are y_{rj}, ($r = 1, 2, \ldots, s$). In this paper, however, we modify the above convention slightly as follows: x_{ij} are regarded as the *system inputs*, not the divisional inputs to the first stage, and y_{rj} are regarded as the *system outputs*, not the divisional outputs from the second stage. On the other hand, the intermediate measures are considered as the *divisional outputs* from the first stage and the *divisional inputs* to the second stage. The reason for this modification will be later clarified in Section 4, where it is shown that the approach of Kao and Hwang (2008) implicitly supports this modified convention.

The input-oriented two-stage network DEA of Kao and Hwang (2008) or the centralized model of Liang et al. (2008) for measuring the efficiency of DMU 0 is given as follows:

$$
\text{(P)}\qquad
\begin{aligned}
\max\quad & uy_o\\
\text{s.t.}\quad & wZ - vX \le 0,\\
& uY - wZ \le 0,\\
& vx_0 = 1,\\
& v,\ u,\ w \ge 0,
\end{aligned}
$$

where $X = (x_{ij}) \in R^{m\times n}$, $Z = (z_{dj}) \in R^{D\times n}$, and $Y = (y_{rj}) \in R^{s\times n}$ are data matrices of inputs, intermediate measures, and outputs, respectively, and v, w, and u are optimization variables of appropriate dimensions representing optimal multipliers on factors. Notice that the same weights (w) are assigned to the intermediate measures no matter whether they are used as inputs or outputs. The optimal objective value (u^*y_0) to model (P) is the system efficiency score (denoted by θ^*), and the first and second stages' efficiency scores are determined by $\theta_1^* = \frac{w^*z_0}{v^*x_0} = w^*z_0$ and $\theta_2^* = \frac{u^*y_0}{w^*z_0}$, respectively, where $*$ denotes an optimal solution to (P). Note also that the system efficiency score is decomposed into a product of the two stages' efficiency scores; $\theta^* = \theta_1^* \cdot \theta_2^* = u^*y_0$.

While Kao and Hwang (2008) and Liang et al. (2008) do not discuss how to obtain frontier projections for inefficient DMUs, Chen et al. (2010) point out that the usual procedure of adjusting the inputs or outputs by the efficiency scores, as in the standard DEA approach, does not necessarily yield a frontier projection. The same argument is also presented in Chen et al. (2013) as a network DEA pitfall. Chen et al. (2010) suggest that the following additional envelopment type of DEA model should be solved to obtain frontier projections for inefficient DMUs:

$$
\text{(D1)}\qquad
\begin{aligned}
\min\quad & \theta\\
\text{s.t.}\quad & X\lambda \le \theta x_o,\\
& Z\lambda \ge \tilde{z}_0,\\
& Z\mu \le \tilde{z}_0\\
& Y\mu \ge y_0,\\
& \lambda,\ \mu \ge 0,
\end{aligned}
$$

where θ, λ, μ, and $\tilde{z}_0$ are optimization variables of appropriate dimensions. Once model (D1) is solved, a frontier projection for DMU 0 is given by $(\theta^*x_0,\ \tilde{z}_0^*,\ y_0)$.

3. Frontier projections

Although Chen et al.'s (2010) procedure outlined in Section 2 is valid, it necessitates solving an additional linear program to obtain frontier projections, which is not the case with the standard DEA. In the standard DEA, duality holds between the multiplier model and the envelopment model, and this makes frontier projections readily obtainable from primal-dual optimal solution pairs. Chen et al.'s (2010) procedure may imply such duality is not readily usable under the two-stage DEA, as also indicated in Chen et al. (2013) as a network DEA pitfall. In this section, however, we show that the duality in the standard DEA naturally migrates to the two-stage network DEA, and develop simple formulas that use primal-dual optimal solution pairs to model (P) to readily determine frontier projections, thereby simplifying and improving the procedure of Chen et al. (2010).

When we solve model (P) using a usual linear program software for DMU 0 under evaluation, we get an optimal primal solution $(v^*,\ w^*,\ u^*)$ as well as a dual optimal solution $(\lambda^*,\ \mu^*,\ \theta^*)$ to the following:

$$
\text{(D)}\qquad
\begin{aligned}
\min\quad & \theta\\
\text{s.t.}\quad & X\lambda \le \theta x_o,\\
& Z\lambda \ge Z\mu,\\
& Y\mu \ge y_0,\\
& \lambda,\ \mu \ge 0.
\end{aligned}
$$

Then, a frontier projection for DMU 0 can be determined as follows:

Frontier projection: $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0) = (X\lambda^*,\ \tilde{z}_0,\ Y\mu^*)$ where $\tilde{z}_0$ is any choice such that $Z\mu^* \le \tilde{z}_0 \le Z\lambda^*$. An easy choice for $\tilde{z}_0$ would be $Z\mu^*$ or $Z\lambda^*$.

In the following proposition, we prove that the above formula truly yields a frontier projection by showing that it attains a system efficiency score of unity and its introduction (to the DMU set) does not move the current frontier.

Proposition 1. *$(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0) = (X\lambda^*,\ \tilde{z}_0,\ Y\mu^*)$ is a frontier projection.*

Proof. We first show that $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0)$ has a system efficiency score of unity. The efficiency of $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0)$ is evaluated by solving the LP

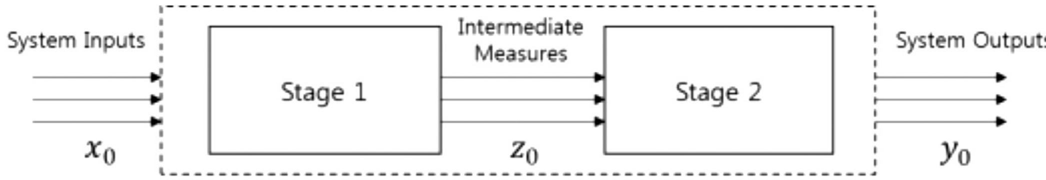

Fig. 1. The generic two-stage process.

problem below:

$$
(\mathrm{P}')\quad
\begin{aligned}
\max\ & u\tilde{y}_0\\
\text{s.t.}\ & wZ - vX \le 0,\\
& uY - wZ \le 0,\\
& w\tilde{z}_0 - v\tilde{x}_0 \le 0,\\
& u\tilde{y}_0 - w\tilde{z}_0 \le 0,\\
& v\tilde{x}_0 = 1.
\end{aligned}
$$

Consider a solution $(\hat{v},\ \hat{w},\ \hat{u}) = (\frac{v^*}{\Gamma}, \frac{w^*}{\Gamma}, \frac{u^*}{\Gamma})$ where $(v^*,\ w^*,\ u^*)$ is an optimal solution to (P) and $\Gamma = v^*X\lambda^* > 0$. Note that $\Gamma = v^*X\lambda^* = \theta^* v^* x_0 = \theta^*$ due to the complementary slackness condition applied to the first constraint of (D). Plugging this solution into the first and second constraints, we get

$$\hat{w}Z - \hat{v}X = \frac{1}{\Gamma}(w^*Z - v^*X) \le 0, \quad \hat{u}Y - \hat{w}Z = \frac{1}{\Gamma}(u^*Y - w^*Z) \le 0.$$

By inserting $(\hat{v},\ \hat{w},\ \hat{u})$ into the third and fourth constraints, we obtain

$$
\begin{aligned}
\hat{w}\tilde{z}_0 - \hat{v}\tilde{x}_0 &= \frac{1}{\Gamma}(w^*\tilde{z}_0 - v^*X\lambda^*) \le \frac{1}{\Gamma}(w^*Z\lambda^* - v^*X\lambda^*)\\
&= \frac{1}{\Gamma}(w^*Z - v^*X)\lambda^* \le 0,\\
\hat{u}\tilde{y}_0 - \hat{w}\tilde{z}_0 &= \frac{1}{\Gamma}(u^*Y\mu^* - w^*\tilde{z}_0) \le \frac{1}{\Gamma}(u^*Y\mu^* - w^*Z\mu^*)\\
&= \frac{1}{\Gamma}(u^*Y - w^*Z)\mu^* \le 0.
\end{aligned}
$$

The fifth constraint with $(\hat{v},\ \hat{w},\ \hat{u})$ becomes

$$\hat{v}\tilde{x}_0 = \frac{1}{\Gamma}(v^*X\lambda^*) = 1.$$

Now we examine the objective function value. Since $\hat{u}\tilde{y}_0 = \frac{1}{\Gamma}u^*Y\mu^* \ge \frac{1}{\Gamma}(u^*y_0) = 1$, it follows that $\hat{u}\tilde{y}_0 \ge 1$. As shown in the above, $\hat{w}\tilde{z}_0 - \hat{v}\tilde{x}_0 \le 0$ and $\hat{u}\tilde{y}_0 - \hat{w}\tilde{z}_0 \le 0$, which leads to $\hat{u}\tilde{y}_0 - \hat{v}\tilde{x}_0 \le 0$. It follows that $\hat{u}\tilde{y}_0 \le 1$ since $\hat{v}\tilde{x}_0 = 1$. Therefore, $\hat{u}\tilde{y}_0 = 1$. These observations confirm that $(\hat{v},\ \hat{w},\ \hat{u})$ is a feasible (and optimal) solution to (P′) and with this multipliers $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0)$ achieves a system efficiency score of unity as claimed.

Next, we show that the original DMUs' efficiency scores do not alter by adding $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0)$ to the data set (i.e., the frontier does not move upwards or downwards). For an original DMU k, its efficiency is evaluated by solving the LP problem below:

$$
(\mathrm{P}'')\quad
\begin{aligned}
\max\ & uy_k\\
\text{s.t.}\ & wZ - vX \le 0,\\
& uY - wZ \le 0,\\
& w\tilde{z}_0 - v\tilde{x}_0 \le 0,\\
& u\tilde{y}_0 - w\tilde{z}_0 \le 0,\\
& vx_k = 1.
\end{aligned}
$$

Let $(v_k^*,\ w_k^*,\ u_k^*)$ be an optimal solution to (P) for DMU k. It suffices to show that $(v_k^*,\ w_k^*,\ u_k^*)$ is still an optimal solution to (P″). For feasibility, we only need to show that $(v_k^*,\ w_k^*,\ u_k^*)$ satisfies the third and fourth constraints. Plugging it into the constraints, we get

$$
\begin{aligned}
w_k^*\tilde{z}_0 - v_k^*\tilde{x}_0 &= w_k^*\tilde{z}_0 - v_k^*X\lambda^* \le w_k^*Z\lambda^* - v_k^*X\lambda^*\\
&= \left(w_k^*Z - v_k^*X\right)\lambda^* \le 0,\\
u_k^*\tilde{y}_0 - w_k^*\tilde{z}_0 &= u_k^*Y\mu^* - w_k^*\tilde{z}_0 \le u_k^*Y\mu^* - w_k^*Z\mu^*\\
&= \left(u_k^*Y - w_k^*Z\right)\mu^* \le 0.
\end{aligned}
$$

The optimality of $(v_k^*,\ w_k^*,\ u_k^*)$ to (P″) is ensured due to the fact that the optimal objective value of (P″) can never be greater than that of (P) which is $u_k^*y_k$.

4. Frontier projection and efficiency score

It is well known that efficiency scores are closely linked with frontier projections under the standard DEA, in which a DMU's input-oriented radial efficiency score can be thought of as how much more inputs the DMU consumes compared with its frontier projection for producing the same amount of outputs. In this section, we demonstrate the same form of link (between frontier projections and efficiency scores) can be established under the two-stage network DEA as under the standard DEA.

Suppose that a frontier projection of a DMU with the two-stage internal structure described in Fig. 1 is determined as $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0)$. Recall our modified convention given in Section 2 that x_0 is regarded as the system inputs (not the divisional inputs to the first stage). Then, the DMU's input-oriented radial system efficiency score θ^* with reference to the frontier projection is $\frac{v^*\tilde{x}_0}{v^*x_0}$, where v^* is an optimal multiplier for the DMU aggregating the multiple system inputs to yield the (one-dimensional) virtual system input. In the same vein, the DMU's input-oriented radial stage 2's divisional efficiency score θ_2^* with reference to the frontier projection is $\frac{w^*\tilde{z}_0}{w^*z_0}$, where w^* is an optimal multiplier for the DMU aggregating the multiple intermediate measures (i.e., divisional inputs to stage 2) to yield the (one-dimensional) virtual stage 2's divisional input. Subsequently, the DMU's input-oriented radial stage 1's divisional efficiency score θ_1^* with reference to the frontier projection is determined as $\frac{\theta^*}{\theta_2^*}$ due to the product-form efficiency decomposition.

With reference to the frontier projection derived in Section 3, we can define system efficiency score and divisional efficiency scores as follows:

1. **System efficiency score**: $\theta^* = \frac{v^*\tilde{x}_0}{v^*x_0} = \frac{v^*X\lambda^*}{v^*x_0} = v^*X\lambda^*$,
2. **Divisional efficiency scores**: $\theta_2^* = \frac{w^*\tilde{z}_0}{w^*z_0}, \theta_1^* = \frac{\theta^*}{\theta_2^*} = \frac{\frac{v^*\tilde{x}_0}{v^*x_0}}{\frac{w^*\tilde{z}_0}{w^*z_0}} = \frac{\frac{v^*\tilde{x}_0}{w^*\tilde{z}_0}}{\frac{v^*x_0}{w^*z_0}}$.

Notice that the system efficiency score calculated by the above formula coincides with the optimal objective value, u^*y_0, of model (P) given in Section 2. This shows that the approach of Kao and Hwang (2008) actually treats x_0 as the system inputs, not the divisional inputs to the first stage, as claimed in Section 2. Given this, it would be interesting to examine the first stage's divisional inputs implied in Kao and Hwang's (2008) model. The formula for the first stage's divisional efficiency score (θ_1^*) implies that the virtual divisional input to the first stage is given by $\frac{v^*x_0}{w^*z_0}$, which can be considered as the virtual system input scaled down by the virtual divisional input to the second stage. In other words, the first stage's divisional efficiency score is calculated by comparing its virtual divisional input $\frac{v^*x_0}{w^*z_0}$ with reference to its frontier projection's virtual divisional input $\frac{v^*\tilde{x}_0}{w^*\tilde{z}_0}$.

Table 1
Data set.

	Operation expenses (X1)	Insurance expenses (X2)	Direct written premiums (Z1)	Reinsurance premiums (Z2)	Underwriting profit (Y1)	Investment profit (Y2)
Taiwan Fire	1,178,744	673,512	7,451,757	856,735	984,143	681,687
Chung Kuo	1,381,822	1,352,755	10,020,274	1,812,894	1,228,502	834,754
Tai Ping	1,177,494	592,790	4,776,548	560,244	293,613	658,428
China Mariners	601,320	594,259	3,174,851	371,863	248,709	177,331
Fubon	6,699,063	3,531,614	37,392,862	1,753,794	7,851,229	3,925,272
Zurich	2,627,707	668,363	9,747,908	952,326	1,713,598	415,058
Taian	1,942,833	1,443,100	10,685,457	643,412	2,239,593	439,039
Ming Tai	3,789,001	1,873,530	17,267,266	1,134,600	3,899,530	622,868
Central	1,567,746	950,432	11,473,162	546,337	1,043,778	264,098
The First	1,303,249	1,298,470	8,210,389	504,528	1,697,941	554,806
Kuo Hua	1,962,448	672,414	7,222,378	643,178	1,486,014	18,259
Union	2,592,790	650,952	9,434,406	1,118,489	1,574,191	909,295
Shingkong	2,609,941	1,368,802	13,921,464	811,343	3,609,236	223,047
South China	1,396,002	988,888	7,396,396	465,509	1,401,200	332,283
Cathay Century	2,184,944	651,063	10,422,297	749,893	3,355,197	555,482
Allianz President	1,211,716	415,071	5,606,013	402,881	854,054	197,947
Newa	1,453,797	1,085,019	7,695,461	342,489	3,144,484	371,984
AIU	757,515	547,997	3,631,484	995,620	692,731	163,927
North America	159,422	182,338	1,141,950	483,291	519,121	46,857
Federal	145,442	53,518	316,829	131,920	355,624	26,537
Royal & Sunalliance	84,171	26,224	225,888	40,542	51,950	6491
Asia	15,993	10,502	52,063	14,574	82,141	4181
AXA	54,693	28,408	245,910	49,864	0.1	18,980
Mitsui Sumitomo	163,297	235,094	476,419	644,816	142,370	16,976

Source: Kao and Hwang (2008).

Table 2
Efficiency scores.

	System efficiency score	Stage 1's divisional efficiency score	Stage 2's divisional efficiency score
Taiwan Fire	0.6992	0.9926	0.7045
Chung Kuo	0.6248	0.9985	0.6257
Tai Ping	0.6900	0.6900	1
China Mariners	0.3042	0.7243	0.4200
Fubon	0.7670	0.8307	0.9233
Zurich	0.3897	0.9606	0.4057
Taian	0.2766	0.6706	0.4124
Ming Tai	0.2752	0.6630	0.4150
Central	0.2233	1	0.2233
The First	0.4660	0.8615	0.5408
Kuo Hua	0.1639	0.6468	0.2534
Union	0.7596	1	0.7596
Shingkong	0.2078	0.6720	0.3093
South China	0.2886	0.6699	0.4309
Cathay Century	0.6138	1	0.6138
Allianz President	0.3202	0.8856	0.3615
Newa	0.3600	0.6276	0.5736
AIU	0.2588	0.7935	0.3262
North America	0.4112	1	0.4112
Federal	0.5465	0.9332	0.5857
Royal & Sunalliance	0.2008	0.7321	0.2743
Asia	0.5895	0.5895	1
AXA	0.4203	0.8426	0.4989
Mitsui Sumitomo	0.1348	0.4287	0.3145

It should also be noted that divisional efficiency scores do not depend on the choice of $\tilde{z}_0$ as proved in the following proposition. In other words, regardless of the choice of $\tilde{z}_0$, we obtain the same divisional efficiency scores.

Proposition 2. *Any choice of $\tilde{z}_0$ satisfying $Z\mu^* \leq \tilde{z}_0 \leq Z\lambda^*$ results in the same value of $w^*\tilde{z}_0$ (i.e., the same divisional efficiency scores).*

Proof. Recall (P′) and its optimal solution $(\hat{v}, \hat{w}, \hat{u})$ given in the proof of Proposition 1. As shown there, $\hat{w}\tilde{z}_0 - \hat{v}\tilde{x}_0 \leq 0$ and $\hat{u}\tilde{y}_0 - \hat{w}\tilde{z}_0 \leq 0$, which lead to $\hat{w}\tilde{z}_0 \leq 1$ and $\hat{w}\tilde{z}_0 \geq 1$. Hence $\hat{w}\tilde{z}_0 = 1$. Since $\hat{w}\tilde{z}_0 = \frac{1}{\Gamma}(w^*\tilde{z}_0)$, it follows that $w^*\tilde{z}_0 = \Gamma = v^*X\lambda^*$, which is not a function $\tilde{z}_0$.

5. Application

As a numerical illustration, we apply our new formulas to the two-stage efficiency assessment of the 24 Taiwanese non-life insurance companies studied in Kao and Hwang (2008). In their study, the production process of the non-life insurance industry is divided into two stages: premium acquisition and profit generation. The premium acquisition stage takes in two inputs, operational expenses and insurance expenses, to yield the intermediate measures, direct written premiums and reinsurance premiums. These intermediate measures are subsequently used by the profit generation stage to produce underwriting profit and investment profit. The data are provided in Table 1.

Table 3
Frontier projections.

	Operation expenses (X1)	Insurance expenses (X2)	Direct written premiums (Z1)	Reinsurance premiums (Z2)	Underwriting profit (Y1)	Investment profit (Y2)
Taiwan Fire	824,219	470,943	5,129,409	673,374	984,143	681,687
Chung Kuo	863,318	628,941	6,287,502	827,782	1,228,502	834,754
Tai Ping	812,495	409,037	4,776,548	560,244	293,613	658,428
China Mariners	182,933	133,022	1,332,365	174,166	248,709	177,331
Fubon	5,138,066	2,708,687	30,127,364	4,177,166	7,851,229	3,925,272
Zurich	1,023,971	260,449	3,807,167	435,394	1,713,598	415,058
Taian	537,342	399,127	3,738,287	654,045	2,239,593	439,039
Ming Tai	1,042,625	515,542	5,553,014	1,009,007	3,899,530	622,868
Central	350,054	212,217	2,166,576	351,793	1,043,778	264,098
The First	607,260	460,202	4,417,507	671,133	1,697,941	554,806
Kuo Hua	191,388	110,225	941,872	263,658	1,486,014	75,639
Union	1,969,433	494,450	7,166,191	849,582	1,574,191	909,295
Shingkong	542,369	284,449	2,649,297	644,400	3,609,236	223,047
South China	402,941	285,432	2,749,750	454,688	1,401,200	332,283
Cathay Century	1,341,182	399,641	5,663,750	634,668	3,355,197	555,482
Allianz President	387,936	132,887	1,899,396	188,103	854,054	197,947
Newa	523,384	390,620	3,504,900	725,272	3,144,484	371,984
AIU	196,078	141,846	1,356,947	224,515	692,731	163,927
North America	65,554	56,849	474,800	108,242	519,121	46,857
Federal	70,816	29,250	302,965	63,960	355,624	26,537
Royal & Sunalliance	16,900	5265	68,296	9611	51,950	6491
Asia	9428	6191	52,063	14,574	82,141	4181
AXA	22,990	11,941	137,690	16,150	8464	18,980
Mitsui Sumitomo	22,014	18,409	159,644	32,943	142,370	16,976

The efficiency results from model (P) are reported in Table 2. The second, third, and fourth columns report the system efficiency scores, the stage 1's divisional efficiency scores, and the stage 2's divisional efficiency scores, respectively. Note that this data set is known to yield unique efficiency decompositions, and thus we obtain the same efficiency results with Kao and Hwang's (2008).

We apply the formula developed in Section 3 to obtain frontier projections for inefficient DMUs. Specifically, we use formula $(\tilde{x}_0, \tilde{z}_0, \tilde{y}_0) = (X\lambda^*, \frac{(Z\lambda^*+Z\mu^*)}{2}, Y\mu^*)$, and the results are reported in Table 3. Note that the frontier projections are equal to what are given in Chen et al. (2010), but are obtained in a much simpler way just using the primal-dual optimal solution pairs.

6. Conclusions

The current study re-examines the issue of deriving frontier projections and divisional efficiency scores by using dual models in the two-stage network DEA model of Kao and Hwang (2008) and Liang et al. (2008). Theory and formulas are developed to calculate the frontier projections and divisional efficiency scores using a set of dual variables.

We should point out that possible multiple optimal solutions exist. Therefore, the frontier projections and divisional efficiency scores are not necessarily unique. In fact, we show that a range of projections for the intermediate measures can be obtained for the frontier projections.

While the current study focus on the assumption of constant returns to scale (CRS), further research is intended for non-CRS situations along with a more general network structures of DMUs. We hope this study is an initial step towards resolving dual model issues in network DEA models as pointed out by Chen et al. (2013).

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