Innovative Applications of O.R.

Piecewise linear output measures in DEA (third revision)

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\textbf{Abstract}

It is assumed in the standard DEA model that the aggregate output (input) is a pure linear function of each output (input). This means, for example, that if DMU \textsubscript{j} generates twice as much of an output as does another DMU \textsubscript{j}, then the former is credited with having created twice as much value. In many situations, however, linear pricing \((\mu, \gamma)\) may not adequately reflect differences in value created from one DMU to another. In this paper, a generalization of the DEA methodology is presented that incorporates piecewise linear functions of factors. We deal specifically with those situations where for certain outputs in an input-oriented model, the weight function \(f(\gamma)\) is described by either a non-increasing or non-decreasing set of multipliers for larger amounts of the factor. We refer to such a variable as exhibiting \textit{diminishing marginal value} (DMV) or \textit{increasing marginal value} (IMV). The DMV/IMV phenomenon is common in many for-profit applications. For example, in the case that \(\gamma\) is the amount of a consumer product generated by DMU \textsubscript{j} and \(\mu\) is the price of that product, it may well be that the market will force lower prices if greater amounts of that product are generated; discounts automatically lead to this DMV situation. Such a phenomenon can arise as well in not-for-profit settings, and we examine such a situation based on earlier work by Cook et al. \cite{Cook1990}. A DEA model for measuring the relative efficiency of highway maintenance patrols. INFOR 28 (2), 113–124.

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1. Introduction

Data envelopment analysis (DEA) as developed by Charnes et al. \cite{Charnes1978} is a tool designed to provide a relative efficiency score for each member of a set of \(n\) decision-making units (DMUs). In the conventional constant return to scale (CRS) model, the efficiency score \(e\) for any DMU "\(\textit{o}\)" is derived by maximizing the ratio of weighted outputs to weighted inputs for that DMU subject to appropriate restrictions. Specifically, if \(Y = (y_i)\) and \(X = (x_i)\) are output and input bundles respectively for DMU \textsubscript{r}, and if \(\mu\) and \(\nu\) denote prices or multipliers of the associated outputs and inputs, then \(e\) is the solution to the nonlinear fractional programming model:

\[
e_o = \max \left(\frac{\sum_t \mu_t y_{t0}}{\sum_t \nu_t x_{t0}}\right)
\]

subject to \(\sum_t \nu_t x_{tj} \leq 1, \quad j = 1, \ldots, n\)

\(\mu_t, \nu_t \geq 0, \quad \text{all } r, i, j\).

It is assumed in (1.1) that the aggregate output (input) is a pure linear function of each output (input). This means, for example, that if DMU \textsubscript{j} generates twice as much of an output as does another DMU \textsubscript{k}, then the former is credited with having created twice as much value. In many situations, however, linear pricing \((\mu, \gamma)\) may not adequately reflect differences in value created from one DMU to another. In the sections to follow, we examine a generalization of the DEA methodology that incorporates piecewise linear functions of factors. We deal specifically with those situations where for certain outputs in an input-oriented model, the weight function \(f(\gamma)\) is described by either a non-increasing or non-decreasing set of multipliers for larger amounts of the factor. We refer to such a variable as exhibiting \textit{diminishing marginal value} (DMV) or \textit{increasing marginal value} (IMV), respectively.

The DMV phenomenon is common in many for-profit applications. In the case that \(\gamma\) is the amount of a consumer product generated by DMU \textsubscript{j} and \(\mu\) is the price of that product, it may well be that the market will force lower prices if greater amounts of that product are generated. Discounts automatically lead to this DMV situation. Such a phenomenon can arise as well in not-for-profit settings, and we examine such a situation based on earlier work by Cook et al. \cite{Cook1990}. In this maintenance patrol setting, we establish that certain factors, previously treated as behaving linearly, should, in fact, be looked upon as having a nonlinear impact on efficiency. Below we justify the relevance of re-examining this application in light of the DMV phenomenon.

In Section 2 we discuss the maintenance patrol setting and give a full description of the factors (variables) to be used in the analysis. Section 3 provides a rationale for viewing certain of the variables as exhibiting DMV behavior. We then develop a modified version of the DEA model to handle such variables. Of similar
importance are IMV situations. As indicated above, in cases involving quality factors, it is often natural to attribute greater incremental value to higher levels of those factors. While we do not specifically develop an IMV example herein, the model structure presented is applicable to both DMV and IMV settings. Section 4 applies the extended model to the maintenance problem and compares the results to the conventional model results. Conclusions and further directions follow in Section 5.

2. Modeling efficiency measurement of highway maintenance patrols

Cook et al. (1990) examined the efficiency of maintenance patrols in the province of Ontario, Canada. Most of the routine maintenance activities on Ontario’s highways fall under the responsibility of the 244 patrols scattered throughout the province. Each patrol is responsible for a fixed number of highway lane-kilometers and oversees the activities associated with that portion of the network. More than 100 different categories of operations/activities exist. They are divided into five areas: surface, shoulder, right-of-way, median, and winter operations.

A popular system for monitoring patrol activities within transportation departments is the maintenance management system (MMS). The MMS is a record keeping system that tracks total work accomplished by the type of operation, patrol, and highway class. All Canadian provinces and US states rely on some form of such a system.

Efficiency evaluation has considerable benefit for highway departments and maintenance units. From the perspective of top management, such evaluation provides a means of distinguishing good managers from less effective ones. Moreover, it can provide an understanding of the impact of such factors as climatic condition, pavement health, and degree of privatization on maintenance effectiveness. In this manner, an efficiency monitoring tool can aid in budget planning and in the design of maintenance policies and practices, and from the point of view of the decision-making unit (the maintenance patrol), particularly the maintenance engineer, routine efficiency evaluation facilities a closer monitoring of how the patrol is conducting its business. The engineer receives an annual status report showing the patrol’s standing relative to other patrols. Furthermore, the tool provides an efficient subset (peer group) of patrols for comparison. Thus, the engineer has a device for evaluating the patrol’s current status and for choosing a direction for future changes.

In Cook et al. (1990), the authors presented a DEA approach for measuring the relative efficiencies of Ontario’s maintenance patrols. That initial model has undergone numerous redesigns, and currently exists in the form of a software package called PAMS (Productivity Analysis of Maintenance System). More recently Cook and Zhu (2003) presented an extension of the original model that addresses the concept of output erosion under decreasing inputs in the input-oriented model.

In the current paper we re-examine the maintenance patrol efficiency measurement problem in light of the discussion in the introduction. Specifically, there are variables in this problem setting, which are of the DMV type, hence properly accommodating the behavior of such variables may provide for a more appropriate treatment of patrol efficiencies. To that end, in the remainder of this section we briefly define the factors (outputs and inputs) used in the evaluation. Full details appear in the Appendix. In the section to follow, a modified version of the conventional DEA model is developed.

It should be pointed out that the research presented herein is largely prompted by the observation of the user that the efficiency standing of patrols appears to be unfairly influenced by the condition rating, particularly at the upper end of the scale (all other variables being the same). Specifically, it is felt that road conditions say at the 80–85 level, should be judged as being nearly as good from the user cost perspective as those in the 90–95 range. This would imply that from the geo-technical point of reference, the impact on the road users in the former class is nearly the same as that of roads in the latter. Thus, there is a need to address this apparent shortcoming in the conventional model structure.

2.1. The factors

In the current setting, the process of selecting factors in a DEA model concentrates on finding effects of maintenance activities together with explanatory or causal factors that allow these effects to be created.

2.2. Outputs

Outputs should measure the effectiveness of the patrols’ actions, or reflect the scale of operations in the patrols. The following four variables are believed to capture the important components of patrol accomplishments.

2.2.1. Size of system (ASF)

The factor is intended to capture the size of the task facing patrol crews. It considers the amount of road surface to be tended, the shoulder and right-of-way area, and winter maintenance requirements.

2.2.2. Average traffic serviced (ATS)

This factor recognizes that greater maintenance efforts may be required on roads with higher traffic. This is true for two reasons. First, larger crew sizes are needed for multilane roads than for lower volume roads. Second, a higher standard of serviceability is often needed on the higher traffic roads. The ATS factor is a function of the average of the daily traffic figures across the highway section managed by the patrol in question.

2.2.3. Accidents (ACC)

Maintenance crews are primarily occupied with the removal of problem areas that could result in accidents (such as washouts or potholes), or with work that results from accidents (such as repairs to damaged guardrails). ACC is a function of several different roadway factors that can influence the occurrence of accidents.

2.2.4. Pavement condition rating (PCR)

Because both maintenance and rehabilitation expenditures are inputs (discussed below), one of their major observable effects is the resulting condition of the pavement. In the case at hand, the pavement condition rating (PCR) happens to measured on a [0,100] scale. Specifically, the model uses a patrol’s average pavement condition rating as a measure of the effectiveness of maintenance and rehabilitation activity.

2.3. Inputs

Inputs here should reflect both the resources consumed at the patrol level as well as the “environment” in which the patrol exists.

2.3.1. Maintenance expenditure (MEX)

This factor includes expenses incurred in-house and those arising from work done by private contractors. This distinction is made because the proportion of privatized work may greatly influence a patrol’s productivity standing. It is also pointed out that if efficiency is being examined in terms of winter maintenance, for example, only that portion of the expenditure figure relating to winter work is used.
2.3.2. Capital (rehabilitation) expenditure (CEX)

Because rehabilitation and maintenance expenditures go hand in hand, the total expenditure on rehabilitation (capital) is an important input. One problem with this factor has to do with when the rehabilitation was conducted. If, for example, maintenance expenditures for a given year are used, the need for these expenditures is, to an extent, a function of the capital work done not only in that year, but as well in several years preceding that. This being the case, capital expenditures for a 5-year period, prior to and including the year in question, were taken in total and used as the rehabilitation budget input.

2.3.3. Climatic factor (CF)

There is unanimous agreement that climatic conditions influence the need for maintenance. Not only do frost heaves necessitate surface work, but as well, snowfall clearly influences winter maintenance activities (such as snow removal and salting). Although no clear relationship has been established between pavement damage and such factors as frost depth, depth of water table, and number of freeze/thaw cycles, it is believed that these and other factors do influence the extent of damage. For the Ontario study, a number of sub-factors were combined to arrive at an overall climatic impact parameter.

3. A piecewise linear DEA model

3.1. Linear versus nonlinear relationships in DEA

The previous section detailed the role played by each of the output and input factors involved in maintenance patrol operations. The conventional DEA approach to evaluating patrol efficiency, and the approach taken previously in Cook et al. (1990), was to treat each such variable as behaving in a linear manner. In this way, the model for efficiency measurement takes the form (1.1).

It may be argued that in a number of situations, certain analysis variables (outputs or inputs) behave in a nonlinear rather than linear manner. In the current application it is worth taking a close look at the variables involved. For purposes of the remainder of this paper, we assume that on the input side, all variables (MEX, CEX and CF) have a linear impact on efficiency. We thus restrict attention to outputs. In the case of the ATS, ASF and ACC variables, the linearity assumption is likely reasonably accurate. For system size (ASF) and accidents (ACC), a linearity assumption would appear to be appropriate in that scaling these values, up or down, would seem to have a proportional affect on budget needs. In the case of traffic, if μATS is to reflect the aggregate value attributable to the number of drivers served by the patrols’ highway network, there is reason to believe that every driver should get an equal weight. At the same time, if doubling the traffic in a patrol would not imply that the budget MEX would need necessarily to be doubled in order for a patrol to maintain its same level of efficiency, this may be an indication that nonlinearity sets in at some traffic level. Thus, ATS may require further study to establish how best to evaluate the accrued value from this factor as higher levels come into being. For present purposes, however, we assume this factor does behave in a conventional linear manner.

A factor that does appear to display a clear nonlinearity in terms of its impact on aggregate output is the PCR. This variable is generally intended to reflect the user cost incurred by drivers. User costs are those costs that are accrued by the user of the facility during the construction, maintenance and/or rehabilitation and everyday use of a roadway section. These costs include vehicle operating costs (tires, gas, oil, etc.) and user delay costs (costs associated with slow downs due to construction and maintenance activities and denial-of-use) (Peterson, 1985). Significant empirical research has been conducted relating to the relationship between vehicle maintenance and operating cost and pavement surface condition. See, for example, Barnes and Langworthy (2003). The latter study, commissioned by the Minnesota D.O.T. attempts to derive estimates of travel costs based on vehicle usage, and to adjust for different driving conditions.

Pavement roughness is an expression of irregularities in the pavement surface that adversely affect a vehicle’s ride quality. Roughness is an important pavement characteristic because it affects not only ride quality but also vehicle operating costs, fuel consumption and maintenance costs. The World Bank found road roughness to be a primary factor in the analyses and trade-offs involving road quality vs. user cost (see Sayers et al., 1986) (see also FHWA, 1989; UMTRI, 1998; Walls and Smith, 1998).

Higher PCR values mean better driving conditions, hence reduced user costs. At the same time, it is well recognized that vehicles driving on pavements in very poor condition, as might be characterized by a PCR of say 40, would experience far more than twice the roughness, hence the user cost, suffered by vehicles on pavements with a PCR of 80. Thus, there is a clear nonlinear relationship between PCR and user cost. The reason we focus here on PCR rather than user cost directly, is that the former is observable (at least in a geo-technical sense), while the latter is not as readily or as reliably available. Fig. 1a provides for a pictorial characterization of this phenomenon. Here, we use saving in user cost as a function of PCR. This function is increasing at a decreasing rate.

In modeling the relationship between pavement condition and user cost, studies like those in Sayers et al. (1986) and Barnes and Langworthy (2003) have resorted to various regression models, including nonlinear approaches. One might view the representation captured in Fig. 1a as a form of production function. If we take this pictorial representation as a true portrayal of the actual relationship under study, and wish to incorporate such into a (linear) DEA structure such as (1.1), it is necessary to invoke some form of linear approximation of the nonlinear pattern. Borrowing from the theory of piecewise linear programming, when a variable exhibits nonlinear behavior, one proceeds by dividing up the scale into K segments, and then assuming that the particular variable behaves linearly in each of those segments. Obviously, the more

![Fig. 1. (a) f(PCR) = savings in user cost versus PCR. (b) Increasing marginal value.](image-url)
segments used the closer will be the piecewise linear approximation to the actual nonlinear function.

Following the above logic, it is reasonable to view the PCR scale as consisting of $K_r$ ranges $[0, L_1], [L_1, L_2], \ldots, [L_{k-1}, L_k]$, where $L_k = 100$. (Recall that the PCR range here is $[0, 100]$). As a working example, we take $K_r = 3$, and use the ranges $[0, 60], [60, 80], [80, 100]$. Clearly, a different choice of ranges may well produce different results in the analysis. And, as indicated, more ranges will provide a closer fit to the nonlinear curve than will be true of fewer ranges. The choice of number and width of ranges would need to be carefully calibrated in an actual case setting, and we do not propose a prescription herein as to the procedures that should be followed in coming up with an appropriate set of such ranges.

Now, let $\mu_{l_k}$ denote the value to be accorded the portion of $y_{l_j}$ that lies in the $k$th range. Specifically, if $y_{l_j} \in [L_{k-1}, L_k)$, then define

$$y_{l_j}^k = \begin{cases} L_k, & k = 1 \\ L_k - L_{k-1}, & k = 2, \ldots, k_j - 1 \\ y_{l_j} - L_{k-1}, & k = k_j \\ 0, & k > k_j \end{cases} \quad (3.1)$$

We may then express the contribution of output $r$ (PCR) to the weighted aggregate of all outputs as

$$\sum_{k=1}^K \mu_{l_k} y_{l_j}^k \quad (3.2)$$

rather than as a single expression $\mu_{l_k} y_{l_j}$ as is typical in conventional DEA applications.

### 3.2. Assurance regions: Intra-variable restrictions

In many DEA settings it is desirable and often essential to impose restrictions on the multipliers $\{\mu_{l_k}\}$ and $\{t_{l_k}\}$. One particularly widely used form of such multipliers is the assurance region (AR) approach of Thompson et al. (1990). This approach enforces constraints on pairs of multipliers (e.g. on output multipliers $\mu_l$):

$$a \leq \frac{\mu_{l_k}}{\mu_{l_1}} \leq b. \quad (3.3)$$

In this paper we examine two levels of restrictions: (1) those relating to the $\{\mu_{l_k}\}_{k=1}^K$, for a given output variable $r$ that displays diminishing marginal value (DMV), as is true of PCR; and (2) those relating to pairs of outputs or inputs, e.g., outputs $r_1$ and $r_2$.

In the first case of *intra-variable* restrictions, it is necessary to capture the idea that the $\{\mu_{l_k}\}_{k=1}^K$, should form a decreasing sequence, and at some desired minimal and maximal rates. While conceptually there is an understanding of the relationship between pavement condition rating and user cost as schematically portrayed in Fig. 1a, there is no definitive functional form available in the literature. As indicated above, various empirical studies have been undertaken, and estimates from sources like Barnes and Langworthy (2003) and elsewhere help to generate ranges for the $\mu_{l_k}$. In the current application as described in Section 4, we impose constraints of the form

$$a_{l_k} \leq \mu_{l_k}/\mu_{l_1} \leq b_{l_k} \quad (3.4)$$

to capture the idea of such ranges. Thus, the multiplier or value according to the PCR in range $k$ is allowed to be a maximum of $b_{l_k}$ times and a minimum of $a_{l_k}$ times the value accorded the PCR in range $k+1$. It is noted that $a_{l_k}$ and $b_{l_k}$ would take on values $>1$ in the DMV case.

It is pointed out that while the pavement maintenance application addressed herein is one in which a variable exhibits DMV, the model structure proposed clearly applies to cases in which variables behave in an IMV manner, as depicted in Fig. 1b. The only tangible difference in modeling the IMV problem would be to set the $a_{l_k}$ and $b_{l_k}$ parameters $<1$.

### 3.3. Inter-variable AR constraints

In the usual DEA setting one encounters *inter-variable* restrictions of the form $(3.4)$. This is appropriate for those cases where diminishing marginal value is not an issue. If one, however, wishes to impose AR constraints linking pairs of variables, one or both of which does exhibit DMV (e.g. PCR), then a problem arises as to how to define $\mu_{l_k}$ for that variable. Consider imposing AR restrictions linking the variables PCR and ASF. As discussed earlier, in the case of the PCR variable, the total value accorded DMU, is given by $\sum_{k=1}^3 \mu_{l_k} y_{l_j}^k$. This is pictured in Fig. 2. In this particular representation, the DMU in question has a PCR level of 85.

Suppose it is believed that the “value” assigned to the PCR variable should be at least 2 times that assigned to the ASF variable. To capture this idea one would normally invoke constraints of the form $(3.5)$. However, it is not clear which of the three PCR multipliers $\{\mu_{l_k}\}_{k=1}^3$ one should select as being representative of that variable. It would appear that the appropriate measure $\mu_{l_k}$ in this case is that derived from the “linearization” of the piecewise linear function as shown in Fig. 3.

**Theorem 3.1.** The linear equivalent of the piecewise linear function $\sum_{k=1}^K \mu_{l_k} y_{l_j}^k$ is given by $f(y_{l_j}) = \mu_{l_k} y_{l_j}$, where $\mu_{l_k}$ is a convex combination of the $\{\mu_{l_k}\}_{k=1}^K$.

**Proof.** If $\mu_{l_k} y_{l_j} = \sum_{k=1}^K \mu_{l_k} y_{l_j}^k$, then $\mu_{l_k} = \sum_{k=1}^K \mu_{l_k} \frac{y_{l_j}^k}{y_{l_j}}$.

Since $\sum_{k=1}^K \frac{y_{l_j}^k}{y_{l_j}} = \frac{y_{l_j}}{y_{l_j}} = 1$, and since $y_{l_j}^k/y_{l_j} > 0$ for all $k$, then the result follows. $\square$

---

Fig. 2. Piecewise linear function of PCR.

Fig. 3. Linearization of PCR function.
Thus, in the case where a variable exhibiting diminishing marginal value is to be compared to a variable for which this is not true, (e.g. PCR compared to ASF), (3.3) would take the form of a set of DMU – specific constraints

\[
a \leq \sum_{k=1}^{K} \mu_{rk} \left( \frac{y_{rj}}{y_{ij}} \right) / \mu_{r+1} \leq b, \quad j = 1, \ldots, n
\]

or

\[
\mu_{r+1} a y_{ij} \leq \sum_{k=1}^{K} \mu_{rk} y_{rj} \leq \mu_{r+1} b y_{ij}, \quad j = 1, \ldots, n.
\]

Each of the constraints in (3.5) creates a form of generalized assurance region (GAR) in the context of Thompson et al. (1990), in that each involves a comparison of the set of multipliers for one DMV variable (PCR in this example) to the single multiplier \( \mu_{r+1} \) for another variable.

Based on the above discussion we thus propose the following piecewise linear DEA (PL-DEA) model for those problem settings in which either DMV or IMV variables are present. We use here \( R_{1, R_{2}} \) to denote the sets of regular and DMV/IMV outputs respectively. \( K_{r} \) represents the number of intervals \( [l_{r-1}, l_{r}] \) defining the piecewise linear function for DMV/IMV variable \( r \), and \( y_{ij}^{r} \) the portion of \( y_{ij} \) that lies in the \( k^{th} \) interval. The parameters \( a_{r1}, b_{r1} \) are the lower and upper bounds on the ratios of pairs of (regular, DMV/IMV) output variables, respectively.

\[
\max \sum_{r \in \mathbb{R}_{1}} \mu_{r} y_{ro} + \sum_{r \in \mathbb{R}_{2}} \mu_{r} y_{r0}^{r} \\
\text{subject to } v_{i} x_{io} = 1 \\
\quad \sum_{r \in \mathbb{R}_{1}} \mu_{r} y_{rj} + \sum_{r \in \mathbb{R}_{2}} \mu_{r} y_{rj}^{r} - \sum_{i \in \mathbb{I}} v_{i} y_{ij} \leq 0, \quad j = 1, \ldots, n \\
\quad \mu_{r+1} a_{r1} \leq \mu_{r} \leq \mu_{r+1} b_{r1}, \quad k = 1, \ldots, K_{r}, \quad r \in \mathbb{R}_{2} \\
\quad \mu_{r+1} a_{r2} y_{rj1} \leq \sum_{k=1}^{K_{r}} \mu_{r} y_{rj}^{r} \leq \mu_{r+1} b_{r2} y_{rj2}, \quad j = 1, \ldots, n, \quad r_{1} \in \mathbb{R}_{1}, r_{2} \in \mathbb{R}_{2} \\
\quad \mu_{r}, v_{i} \geq 0.
\]

In case it is desired to invoke AR restrictions linking two variables \( r \) and \( r+1 \), with each exhibiting DMV/IMV, (for example, PCR compared to traffic), then (3.5) becomes

\[
a \leq \sum_{k=1}^{K_{r}} \mu_{rk} \left( \frac{y_{rj}}{y_{ij}} \right) / \sum_{k=1}^{K_{r+1}} \mu_{r+1,k} \left( \frac{y_{r+1,j}}{y_{i,j}} \right) \leq b.
\]

Here, \( K_{1}, K_{2} \) denote the numbers of intervals defining the piecewise linear arrangements in variables \( r \) and \( r+1 \), respectively.

It is noted that (3.6) is a form of Cone Ratio DEA structure (Charnes et al., 1990). In particular, it is the conventional (CCR) DEA model with a set of assurance region constraints as per Thompson et al. (1990), as well as a set of linear constraints on the output multipliers (the cone ratio). Clearly, one might reasonably view the modified DEA model (3.6) as simply a constrained DEA structure in which a single (DMV/IMV) variable \( r \) has been replaced by a set of \( K_{r} \) variables. This is in fact the case, allowing of course for the provision that the multipliers \( \mu_{r} \) assigned to those variables form a decreasing (increasing for IMV variables) sequence. This having been said, the model does, however, possess the important feature that it can accommodate say quality measurement issues inherent in many problems; a capability that is not immediately available in the conventional CRS and VRS models.

In the section to follow we apply these ideas to examine the efficiency of maintenance patrols in the setting where we need to account for a DMV variable.

4. Application of the PL-DEA model to the maintenance patrol problem

Consider the situation where we wish to evaluate the efficiency of a set of 20 maintenance patrols in terms of the set of variables described earlier. Table 1 displays the data for these patrols. In evaluating the efficiency of patrols, two types of analyses were carried out. In the first, we treat the PCR as behaving linearly and run the usual DEA model (1.1) with appropriate AR restrictions. Specifically, AR constraints are imposed on the multipliers connecting the PCR and ASF variables, namely

\[
2 \leq \mu_{4}/\mu_{1} \leq 4 \quad \text{or} \quad 2\mu_{1} \leq \mu_{4} \leq 4\mu_{1}.
\]

(4.1)

Efficiency scores from applying (1.1) in the presence of (4.1) appear in the second column of Table 2 under the heading “Standard Efficiency”.

The second level of analysis views the contribution of the PCR to the overall output as a piecewise linear function of that variable. For example, the PCR of 85 for DMU #4 will now be represented (and replaced in the DEA analysis) by the three variables we denote

<table>
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<th>Crew number</th>
<th>Outputs</th>
<th>Inputs</th>
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Table 2: Efficiency results (1)

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<th>DMU #</th>
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as PCR1, PCR2 and PCR3 (60, 20, 5). We denote the corresponding multipliers by $\mu_41, \mu_42, \mu_43$, and replace the term $\mu_41y_4$ in model (1.1) by the piecewise term $\sum_{k=1}^{6} \mu_41^k y_4^k$ in (3.6). Two types of AR restrictions are applied in this setting, namely:

Type 1: Intra-variable constraints: We impose on the multipliers $\mu_41, \mu_42, \mu_43$, the constraints

$$2\mu_42 \leq \mu_41 \leq 4\mu_42$$ and $$2\mu_43 \leq \mu_42 \leq 3\mu_43.$$ (4.2)

to invoke the requirement that the PCR should display the DMV behavior discussed earlier. Specifically, we require that the impact of that portion of the PCR in the first range, namely $y_41$, should be at least twice as much and not more than four times as much as that portion in the second range. A similar restriction is imposed on the multipliers of $y_42$ and $y_43$. The choice of parameters (2, 4) and (2, 3) used in (4.2) is purely arbitrary for purposes of demonstration, and would emphasize that in an actual situation this choice would need to be made with care.

Type 2: Inter-variable constraints: Here we impose a set of DMU-specific constraints linking the importance attached to the PCR to the variable ASF that reflect (3.5). For DMU #4, for example, (3.5) takes the form

$$2\mu_41 \leq (60\mu_41 + 20\mu_42 + 5\mu_43)/85 \leq 4\mu_41.$$ The results from the DEA analysis, using model (3.5) are displayed in the last column of Table 2 under the heading “Piecewise Efficiency”. It is noted that for the particular data used, many of the efficiency scores stay approximately the same under the two analyses. In particular, the DMUs making up the efficient frontier are the same under the two analyses. Generally, the efficiency scores resulting from the piecewise linear DEA analysis can be either higher or lower than their “standard” DEA counterparts. In the particular analysis carried out herein most scores decreased under the piecewise model relative to their standing in the standard model. Specifically, only DMUs 9 and 13 experienced a slight increase in efficiency under the more general piecewise model. It is also the case that only two pairs of DMUs reversed their relative orderings. DMUs 3 and 7 were ranked 3 < 7 (i.e. 7 was ranked higher than 3) in the standard model. This was reversed in the piecewise model. The same is true of DMU 4 versus 13. All other pairs were relatively ranked the same under both models.

4.1. An alternative set of bounds

The change in efficiencies in going from the standard to the piecewise model is largely a function of the bounds imposed on the variables. To demonstrate this point, reconsider the two analyses, but this time replacing the constraints (4.1) in the standard model by

$$5\mu_1 \leq \mu_4 \leq 7\mu_1.$$ (4.3)

and by replacing (4.2) by

$$4\mu_42 \leq \mu_41 \leq 8\mu_42$$ and $$4\mu_43 \leq \mu_42 \leq 8\mu_43.$$ (4.4)

in the piecewise model. In regard to (3.5) for the piecewise model, we set $a = 5$, and $b = 7$ (consistent with (4.3)). As well, for both models we apply AR constraints on $\mu_1, \mu_2, \mu_3$, specifically

$$\mu_1 \geq 3\mu_2$$ and $$\mu_3 \geq 3\mu_2.$$ (4.5)

With the above imposed restrictions, multipliers are more strictly controlled than under the first analysis. As shown by the results in Table 3, there are now substantial differences between the outcomes from the two models, both in terms of scores and rank positions of the DMUs. In a number of situations, it is the case that maintenance patrols with lower PCRs that were penalized with low efficiency ratings under the standard analysis, have been some-what compensated for this, with boosted ratings under the PL-DEA approach. This is the case for DMUs 1, 2 and 11 whose PCRs are at or below 70. Conversely, a number of DMUs with higher PCRs (at or above 85), such as 4, 5, 7, 14, 18 and 20, witnessed a drop in their scores in moving from the standard DEA to the PL-DEA approach. In the latter, one can argue that these patrols were perhaps being overcompensated in the standard analysis, and credited with a greater accomplishment than deserved. Given that efficiency scores are a key driver in setting budgets, the PL-DEA model provides a more fair description of performance, resulting in budget allocations that are more in line with needs.

There are, of course, examples such as DMU #19, where the PCR is low, yet the efficiency is the same under both models (100% in this case). This generally occurs because the other factors, both outputs and inputs, come into play. However, one can generally say that, all other things being equal, the PL-DEA approach raises the scores (arising from the standard model) of DMUs with low PCR values, and reduces the scores for those DMUs with high PCRs, more correctly reflecting the underlying user cost implications.

Statistically, the Spearman Correlation coefficients under both analyses indicate significant rank correlation between the two sets of ranks. ($R = .98$ under the first analysis and $.75$ under the second, as compared to the critical 1% value of .534). One might tend to conclude from this that there is little difference in the resulting rankings of the DMUs under the two models. However, as illustrated by the last column of Table 3, the ranks are clearly very different under the two approaches in the second analysis, contrary to the Spearman statistics.

If one accepts that the piecewise approach is the proper way of treating the PCR variable here, then the results from the second analysis would appear to more accurately reflect reality than is true of the results from the conventional analysis.

4.2. An unrestricted analysis

One might look at the three new variables (used to capture the piecewise linearity) as independent outputs, and, therefore, model them without the imposition of the intra-variable AR constraints. This was done and the results, along with those of Table 2, are shown in Table 4. It is noted that in addition to DEA scores being different from those of the standard and piecewise models, the ranks differ substantially. Here the Spearman rank Correlation Coefficient is at 87%, when we compare the ranks of the DMU
Appendix A. Description of analysis factors

A.1. ASF – Area served factor

This factor was chosen to measure the extent of the work load for which the patrol has responsibility. The ASF factor value is calculated from the formula

$$ASF = \sum i L_i (TLE_i (A_{j} + C) + L_i (S_i B_j + D)),$$

where $L_i$ – length of road section $i$; $TLE_i$ – two-lane equivalent of road section $i$; $S_i$ – shoulder width of road section $i$; $A_{j}$ – coefficient for road surface type $j$ (the one in road section $i$); $B_j$ – coefficient for shoulder type $j$ (the one in road section $i$); $C$ – coefficient for winter operations; $D$ – coefficient for other operations (ROW, median, etc.).

A.2. ATS – Average traffic served

This factor is intended to be a measure of the overall benefit to the users of the highway system in a patrol. The formula for computing ATS is given by

$$ATS = 10^{-3} \sum i L_i (A_{j} D_i),$$

where $A_{j} D_i$ is the annual average daily traffic and $10^{-3}$ is a scaling factor designed to bring ATS within a reasonable range for analysis.

A.3. PCR – pavement condition rating

This factor measures the extent of damage to the road surface. It is generally a reflection of the amount of cracking, ruts and roughness. Geo-technical practices differ from one jurisdiction to another as to how the various roadway distresses are measured and included in the rating.

A.4. APF – accident prevention factor

Much of the work of maintenance staff arises due to the need to prevent accidents (surface and shoulder repairs, washouts, etc.) In this regard, accident prevention can be viewed as a cause or goal of maintenance.

A reasonable measure of accident prevention should be directly proportional to traffic level (ATS), and inversely proportional to the observed number of accidents. The chosen form is given by

$$APF = \frac{100 \times ATS}{C},$$

where 100 is a scaling factor and $C$ is the number of road accidents, during the observed period, on all road sections serviced by a patrol.

A.5. MEX – Maintenance expenditures

This is the total of all expenditures linked to the patrol. It includes both “in-house” work as well as maintenance activities performed by private contractors. Moreover, MEX includes any district-supplied services such as equipment and district supervisors’ salaries.

A.6. CEX – Capital expenditures

This is the total of all capital expenditures made toward improving the existing highway infrastructure. This would include resurfacing, shoulder paving, repairs to structures, dome construction, etc. – all activities which complement maintenance efforts. Excluded are new link and new structure construction, since these do not directly complement maintenance.

5. Conclusions

In this paper we have examined the DEA structure in the presence of output variables that do not exhibit linear behavior, but rather experience either diminishing marginal value (DMV) or increasing marginal value (IMV). This DMV/IMV concept gives rise to the need to replace the linear form of such variables by a more appropriate nonlinear representation. A piecewise linear function is often a reasonable approximation of such nonlinear behavior. We demonstrate that in the presence of such piecewise linear variables, a modified form of the DEA structure results, provided that appropriate assurance region (AR) restrictions are imposed. This modified DEA model is applied to the maintenance patrol setting examined earlier by Cook et al. (1990). This application serves to illustrate that efficiency scores can increase, decrease or remain the same, in comparison to those obtained from the standard DEA model. We show that the extent of this change can be greatly influenced by the nature of the AR constraints applied.

This line of research can have important implications, in that many real world problems are characterized by variables that exhibit such nonlinear behavior, meaning that the standard linear structures do not appropriately reflect the reality of those situations. The proposed model provides a means of dealing with these problems.

Acknowledgements

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A.7. CLF – Climatic factor

What can often be an overriding consideration in the performance of a patrol is the environmental circumstances in which that patrol must operate. The amount of snowfall, for example, will clearly influence the level of winter maintenance (snow removal and salting) needed. The extent of spring breakups will directly influence the need for summer road surface work.

Four sub-factors were taken into account in arriving at an overall climatic factor, namely snowfall, major temperature cycles, minor temperature cycles and rainfall.

Available data from weather stations were used to compute these sub-factors.

The overall climatic factor for a patrol is computed from:

$$\text{CLF}_k = \sum_i P_{ki} \left( \sum_j (W_j / D_{ij}) \right)$$

where \( k \) – patrol index; \( P_{ki} \) – weight of station \( i \) in calculating the climatic factor of patrol \( k \); \( W_j \) relative importance weight of climatic factor \( j \).

\( W_1 = 50, \ W_2 = 300, \ W_3 = 20,000, \ W_4 = 1000 \).

It is noted that the weights \( W_j \) were chosen while taking into account the numerical scales of each of the climatic factors (e.g. the snowfall numbers are much greater in size than the major cycle numbers). In addition, the weights were selected with attention to the resultant CLF measure being relatively of the same order of magnitude as the other efficiency factors.

References


Federal Highway Administration (FHWA), 1989. Federal-Aid Highway Program Manual, Federal Highway Administration, 6, Ch. 2, Section 4, Subsection 1, Federal Highway Administration. Washington, DC.


