INFEASIBILITY OF SUPER-EFFICIENCY DATA ENVELOPMENT ANALYSIS MODELS[1]

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ABSTRACT

The paper investigates the infeasibility of super-efficiency data envelopment analysis (DEA) models in which the unit under evaluation is excluded from the reference set. Necessary and sufficient conditions are provided for infeasibility of the super-efficiency DEA measures. By the returns to scale (RTS) classifications obtained from the standard DEA model, we can further locate the position of the unit under evaluation when infeasibility occurs. It is shown that the ranking of the total set of efficient DMUs is impossible because of the infeasibility of super-efficiency DEA models. Also we are able to identify the endpoint positions of the extreme efficient units. The results are useful for sensitivity analysis of efficiency classifications.

Key words: Data envelopment analysis (DEA); efficient; infeasibility; returns to scale (RTS); super-efficiency.

RÉSUMÉ:

Cet article étudie la non-faisabilité des modèles super-efficacite d'analyse d'enveloppement de données (DEA), pour lesquels l'unité en cours d'évaluation est exclue de l'ensemble de référence. Des conditions nécessaires et suffisantes sont données pour la non-faisabilité des mesures de super-efficacité DEA. La classification des rendements d'echelle obtenue à partir du modèle DEA standard nous permet de localiser la position de l'unité en cours d'évaluation quand la non-faisabilité se produit. Nous demontrons que tous les rangs de l'ensemble des DMUs efficaces ne sont pas disponibles en raison de la non-faisabilité des modeles super-efficacite DEA. Nous identifions egalement les positions extrêmes des unités extrêmement efficaces. Les résultats sont utiles pour les analyses de sensibilité des classifications d'efficacité.

Mots-clés : Analyse d'envelopment de données (DEA); efficace; non-faisabilité; rendements d'échelle; super-efficacité

1. INTRODUCTION

Over the past twenty years, data envelopment analysis (DEA) has been one of the fastest growing areas of interest in management science (Seiford, 1996, 1997). One frequent use of DEA is in determining the relative efficiencies of a set of decision making units (DMUs) consuming multiple inputs to produce multiple outputs. A DMU is said to be efficient if its performance relative to other DMUs from the sample can not be improved. Banker, Charnes and Cooper (BCC) (1984) showed that the original DEA model by Charnes, Cooper and Rhodes (CCR) (1978) can be regarded as a mixture of a technical efficiency measure and a scale efficiency measure. The latter measure is relative to the economic notion of returns to scale (RTS). Thus the CCR and the BCC models can be used to determine the RTS classifications – increasing, constant and decreasing returns to scale (IRS, CRS, and DRS) for each of the DMUs. (See Banker and Thrall,

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1992.) By appending additional constraints on the intensity vector of the CCR model, one obtains other DEA models for investigating RTS (see, Färe, Grosskopf and Lovell, 1994.) Seiford and Thrall (1990) provided these basic DEA models in input-based and output-based versions.

In recent years, variants of the basic DEA models – super-efficiency DEA models – have appeared in the literature. These super-efficiency DEA models in which the DMU under evaluation is not included in the reference set, can play important roles in various situations. Charnes *et al.* (1992) and Zhu (1996) used them to study the sensitivity of the efficiency classifications (see also Charnes *et al.* (1996) and Seiford and Zhu (1998a,b)). Andersen and Petersen (1993) proposed their use in a procedure for ranking the efficient DMUs. Färe, Grosskopf and Lovell (1994) employed them to measure technology and productivity changes. Also, the super-efficiency DEA models can be used in two-person ratio efficiency games (Rousseau and Semple, 1995) and in detecting influential observations (Wilson, 1995) and in identifying the extreme efficient DMUs (Thrall, 1996).

However, as noted by Thrall (1996), the super-efficiency CCR model may be infeasible. Furthermore, as shown in Zhu (1996), the super-efficiency CCR model is infeasible if and only if certain zero patterns appear in the data domain. As a matter of fact, other super-efficiency DEA models, may also be infeasible even when such zero patterns are not present in the input/output data. Since the super-efficiency DEA models now are widely used in a variety of ways, the study of the feasibility of these models has become crucial importance. The current paper provides necessary and sufficient conditions for infeasibility of the super-efficiency DEA models. We further investigate the infeasibility by using the RTS classifications obtained from the original DEA models. It is shown that the infeasibility information is useful for locating the endpoint positions of the extreme efficient DMUs. The infeasibility information can also be used in a sensitivity analysis of the efficiency classifications under different DEA models.

The paper unfolds as follows. Section 2 provides the super-efficiency DEA models. Assumptions and some basic relationships are described. Section 3 develops necessary and sufficient conditions for the infeasibility of the super-efficiency DEA models. Concluding remarks are given in section 4.

2. SUPER-EFFICIENCY DEA MODELS

We assume that there are n DMUs. Each DMU_j $(j = 1, 2, \ldots, n)$ consumes a vector of inputs, x_j, to produce a vector of outputs, y_j. On the basis of the basic DEA models provided in Seiford and Thrall (1990), the super-efficiency DEA model can be expressed as

$$\begin{array}{ll}
\textit{output-based} & \textit{input-based} \\
\max \varphi & \min \rho \\
\text{s.t. } \sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j x_j \leq x_0; & \text{s.t. } \sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j x_j \leq \rho x_0; \\
\sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j y_j \geq \varphi y_0; & \sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j y_j \geq y_0; \\
\varphi, \lambda_j \geq 0, j \neq 0; & \rho, \lambda_j \geq 0, j \neq 0;
\end{array}$$

For SE-CCR append *nothing.*

For SE-BCC append $\sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j = 1.$

$$\text{For SE-NIRS append } \sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j \leq 1.$$

$$\text{For SE-NDRS append } \sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j \geq 1.$$

where (x_0, y_0) represents DMU_0. SE-CCR represents the super-efficiency CCR model which assumes constant returns to scale (CRS). SE-BCC represents the super-efficiency BCC model in which increasing, constant and decreasing return to scale (IRS, CRS and DRS) are allowed, because of the convexity constraint on the (intensity) lambda variables. SE-NIRS and SE-NDRS represent the two super-efficiency DEA models satisfying nonincreasing and nondecreasing returns to scale (NIRS and NDRS) respectively.

We see that the difference between the super-efficiency and the original DEA models is that the DMU_0 under evaluation is excluded from the reference set. *I.e.*, the super-efficiency DEA models are based on a reference technology constructed from all other DMUs.

As in Charnes, Cooper and Thrall (1991), the DMUs can be partitioned into four classes E, E', F and N described as follows. First, E is the set of extreme efficient DMUs. Second, E' is the set of efficient DMUs that are not extreme points. The DMUs in set E' can be expressed as linear combinations of the DMUs in set E. Third, F is the set of frontier points (DMUs) with non-zero slack(s). The DMUs in set F are usually called weakly efficient. Fourth, N is the set of inefficient DMUs.

Thus if a specific $DMU_0 \in E'$, F or N and is not included in the reference set, then the efficient frontiers (constructed by the DMUs in set E) remain unchanged. As a result, the super-efficiency DEA models are always feasible and equivalent to the original DEA models when $DMU_0 \in E'$, F or N. Thus we only need to consider the situation when $DMU_0 \in E$.

Thrall (1996) showed that if the super-efficiency CCR model, SE-CCR model, is infeasible, then $DMU_0 \in E$. However, he failed to recognize that the output-based SE-CCR model is always feasible for the trivial solution which has all variables set equal to zero. Moreover, Zhu (1996) showed that the input-based SE-CCR model is infeasible if and only if a certain pattern of zero data occurs in the inputs and outputs. *E.g.*, DMU_0 has some zero inputs which are positive for all other DMUs or DMU_0 has some positive outputs which are equal to zero for all other DMUs. Therefore we study the infeasibility of the other super-efficiency DEA models, where we may assume that all data are positive[2].

From the convexity constraint $\left(\sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j = 1\right)$ on the intensity lambda variables, we immediately have:

Proposition 1

$DMU_0 \in E$ under the BCC model if and only if *$DMU_0 \in E$ under the NIRS model or NDRS model.*

Thus in the discussion to follow, we limit our consideration to *$DMU_0 \in E$ under the BCC model.* Moreover by Thrall (1996), we have

Proposition 2

Let φ^* and ρ^* denote, respectively, optimal values to the output-based and input-based super-efficiency DEA models when evaluating an extreme efficient DMU_0, then

[2]Note that the BCC model is translation invariant when DMU_0 is efficient (see Ali and Seiford, 1990)

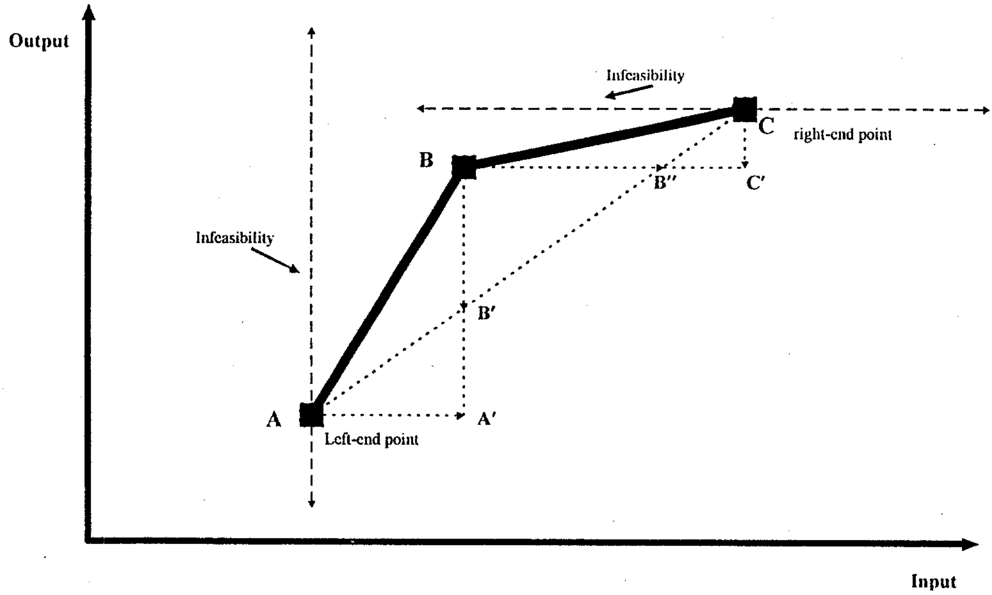

Figure 1: Super-efficiency BPP Model

(a) Either $\varphi^* < 1$ *or the specific output-based super-efficiency DEA model is infeasible;*

(b) Either $\rho^* > 1$ *or the specific input-based super-efficiency DEA model is infeasible.*

Proof

Since $DMU_0 \in E$ under the BCC model, by Thrall (1996, Theorem 3, p. 117) and Proposition 1, this proposition is true. •

Figure 1 illustrates how SE-BCC works and the infeasibility for the case of a single output and a single input case. We have three BCC-extreme-efficient DMUs, A, B and C. The BCC efficient frontier is ABC where AB exhibits increasing returns to scale (IRS) and BC exhibits decreasing returns to scale (DRS). SE-BCC model evaluates point B by reference to B' and B'' on section AC through output-reduction and input-increment, respectively. In an input-based SE-BCC model, point A is evaluated against A', however, there is no referent DMU for point C for input variation. Therefore, the input-based SE-BCC model is infeasible at point C. Similarly, in an output-based SE-BCC model, point C is evaluated against C', however, there is no referent DMU for point A for output variation. Therefore, the output-based SE-BCC model is infeasible at point A. Note that point A is the left most end point and point B is the right most end point this frontier.

This simple example indicates that the ranking of the total set of efficient DMUs is impossible because of the infeasibility of super-efficiency DEA models. In the next section, we will (1) develop the necessary and sufficient conditions for the infeasibility of various super-efficiency DEA models in a multiple inputs and multiple outputs situation, and (2) reveal the relationship between infeasibility and returns to scale classification. (Note that, in Figure 1, point A is associated with IRS and point C is associated with DRS).

3. INFEASIBILITY OF SUPER-EFFICIENCY DEA MODELS

3.1 Output-based SE-BCC Model

Theorem 1:

For a specific extreme efficient $DMU_0 = (x_0, y_0)$, the output-based SE-BCC model is infeasible if and only if *$(x_0, \delta y_0)$ is efficient under the original BCC model for any $0 < \delta \leq 1$.*

Proof

Suppose that the output-based SE-BCC model is infeasible and that $(x_0, \delta^0 y_0)$ is inefficient, where $0 < \delta^0 \leq 1$. Then

$$\begin{aligned} \varphi_0^* &= \max \varphi_0 \\ \text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_j &\leq x_0; \\ \sum_{j=1}^{n} \lambda_j y_j &\geq \varphi_0(\delta^0 y_0); \\ \sum_{j=1}^{n} \lambda_j &= 1. \end{aligned} \tag{1}$$

has a solution of λ_j^* $(j \neq 0)$, $\lambda_0^* = 0$, $\varphi_0^* > 1$. Since $\lambda_0^* = 0$, we have (1) is equivalent to an output-based SE-BCC model and thus the output-based SE-BCC model is feasible, a contradiction. This completes the proof of the *only if* part.

To establish the *if* part, we note that if the output-based SE-BCC model is feasible, then $\varphi^* < 1$ is the maximum radial reduction of all outputs preserving the efficiency of DMU_0. Therefore δ can not be less than φ^*, otherwise DMU_0 will be inefficient under the original output-based BCC model. Thus the output-based SE-BCC model is infeasible. •

Theorem 2

The output-based SE-BCC model is infeasible if and only if *$\hbar^* > 1$, where $\hbar^*$ is the optimal value to (2).*

$$\begin{aligned} \hbar^* &= \min \hbar \\ \text{s.t.} \quad \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_j &\leq \hbar x_0; \\ \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j &= 1 \\ \lambda_j &\geq 0, \quad j \neq 0. \end{aligned} \tag{2}$$

Proof

We note that for any λ_j $(j \neq 0)$ with $\sum_{j=1, j\neq 0}^{n} \lambda_j = 1$, the constraint $\sum_{j=1, j\neq 0}^{n} \lambda_j y_j \geq \varphi y_0$ always holds[3]. Thus the output-based SE-BCC is infeasible if and only if there exists no λ_j $(j \neq 0)$ with $\sum_{j=1, j\neq 0}^{n} \lambda_j = 1$ such that $\sum_{j=1, j\neq 0}^{n} \lambda_j x_j \leq x_0$ holds. This means that the optimal value to (2) is greater than one, *i.e.*, $\hbar > 1$. •

[3]Recall that we assume all data are positive.

3.2 Infeasibility and Returns to Scale

Moreover, note that the DMU_0 is also CCR efficient if and only if CRS prevail. Therefore, if IRS or DRS prevail, then DMU_0 must be CCR inefficient. Thus, in this situation, the SE-CCR model is identical to the original CCR model. By Banker and Thrall (1992), IRS or DRS on DMU_0 can be determined by

Lemma 1

The returns to scale (RTS) for DMU_0 can be identified as IRS if and only if

$\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j^* < 1$ *in all optima for the SE-CCR model and DRS* if and only if

$\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j^* > 1$ *in all optima for the SE-CCR model.*

Lemma 2

If DMU_0 exhibits DRS, then the output-based SE-BCC model is feasible. Moreover $\varphi^* < 1$, where φ^* is the optimal value to the output-based SE-BCC model.

Proof

The output-based SE-BCC model is as follows:

$$\begin{aligned} \varphi^* &= \max \varphi \\ \text{s.t.} \quad \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_j &\leq x_0; \\ \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_j &\geq \varphi y_0; \\ \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j &= 1; \\ \varphi,\ \lambda_j &\geq 0, \quad j \neq 0. \end{aligned} \tag{3}$$

Let $\theta = 1/\varphi$. Multiplying all constraints in (3) by θ yields:

$$\begin{aligned} &\min \theta \\ \text{s.t.} \quad \sum_{\substack{j=1 \\ j\neq 0}}^{n} \tilde{\lambda}_j x_j &\leq \theta x_0; \\ \sum_{\substack{j=1 \\ j\neq 0}}^{n} \tilde{\lambda}_j y_j &\geq y_0; \\ \sum_{\substack{j=1 \\ j\neq 0}}^{n} \tilde{\lambda}_j = \theta &= \frac{1}{\varphi}; \\ \varphi,\ \theta,\ \tilde{\lambda}_j &\geq 0, \quad j \neq 0. \end{aligned} \tag{4}$$

where $\tilde{\lambda}_j = \theta \lambda_j$ $(j \neq 0)$.

Since DMU_0 exhibits DRS, then by Lemma 1, $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j^* > 1$ in all optima to the following SE-CCR model:

$$\begin{aligned} &\min \rho \\ \text{s.t.} \quad &\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j x_j \leq \rho x_0; \\ &\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j y_j \geq y_0; \\ &\rho,\ \lambda_j \geq 0. \end{aligned} \tag{5}$$

Let $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j^* = \theta$. Obviously $\theta > p$ is a feasible solution to (5). This in turn indicates that λ_j^* $(j \neq 0)$ and θ is a feasible solution to (4). Therefore (3) is feasible. Furthermore by Proposition 2, we have that $\varphi^* < 1$, where φ^* is the optimal value to (3). •

Theorem 3

If the output-based SE-BCC is infeasible, then DMU_0 exhibits IRS or CRS.

Proof

Suppose that DMU_0 exhibits DRS. By Lemma 2, the output-based SE-BCC is feasible which leads to a contradiction. •

Theorems 1 and 2 indicate that if the output-based SE-BCC model is infeasible, then is one of the *endpoints*. Moreover if IRS prevail, then DMU_0 is a *left endpoint* (see Figure 1).

3.3 Other Output-based Super-efficiency Models

Now consider the output-based SE-NIRS and SE-NDRS models. Obviously, we have a feasible solution of $\lambda_j = 0$ $(j \neq 0)$ and $\varphi = 0$ in the output-based SE-NIRS model. Therefore, we have

Theorem 4

The output-based SE-NIRS model is always feasible.

Lemma 3

The output-based SE-NDRS model is infeasible if and only if *the output-based SE-BCC model is infeasible.*

Proof

The *only if* part is obvious and hence is omitted. To establish the *if* part, we suppose that the output-based SE-NDRS model is feasible. *I.e.*, we have a feasible solution with $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j \geq 1$ for the output-based SE-NDRS model. If $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j = 1$, then this solution is also feasible for the output-based SE-BCC. If $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j > 1$, let $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j = d > 1$. Then $\sum_{\substack{j=1\\j\neq 0}}^{n} \tilde{\lambda}_j x_j \leq \sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j x_j \leq x_0$, where $\tilde{\lambda}_j = \lambda_j / d$ $(j \neq 0)$ and $\sum_{\substack{j=1\\j\neq 0}}^{n} \tilde{\lambda}_j = 1$. Therefore $\tilde{\lambda}_j$ $(j \neq 0)$ is a feasible solution to the output-based SE-BCC model. Both possible cases lead to a contradiction. Thus, the output-based SE-NDRS model is infeasible if the output-based SE-BCC model is infeasible. •

On the basis of this lemma, we obtain

Theorem 5

For a specific extreme efficient $DMU_0 = (x_0, y_0)$, we have:

(a) *The output-based SE-NDRS model is infeasible* if and only if, $(x_0, \delta y_0)$ *is efficient under the original BCC model for any* $0 < \delta \leq 1$;

(b) *The output-based SE-NDRS model is infeasible* if and only if $\hbar^* > 1$, *where* $\hbar^*$ *is the optimal value to (2).*

Remark

If $DMU_0 \in E$ for the NDRS model, then DMU_0 exhibits IRS or CRS. By Proposition 1, DMU_0 also lies on the BCC frontiers that satisfy IRS or CRS. *I.e.*, the original BCC and NDRS models are identical for DMU_0. Thus $(x_0, \delta y_0)$ is also efficient for the original NDRS model for any $0 < \delta \leq 1$.

3.4 Input-based SE-BCC Model

Now we consider the input-based super-efficiency DEA models.

Theorem 6

For a specific extreme efficient $DMU_0 = (x_0, y_0)$, *the input-based SE-BCC model is infeasible* if and only if $(\chi x_0, y_0)$ *is efficient under the original BCC model for any* $1 \leq \chi < +\infty$.

Proof

Suppose the input-based SE-BCC model is infeasible and assume that $(\chi^0 x_0, y_0)$ is inefficient, where $1 \leq \chi^0 < +\infty$. Then

$$\begin{aligned} \rho_0^* &= \min \rho_0 \\ \text{s.t.} \sum_{j=1}^{n} \lambda_j x_j &\leq \rho_0(\chi^0 x_0); \\ \sum_{j=1}^{n} \lambda_j y_j &\geq y_0; \\ \sum_{j=1}^{n} \lambda_j &= 1. \end{aligned} \tag{6}$$

has a solution of λ_j^* $(j \neq 0)$, $\lambda_0^* = 0$, $\rho_0^* < 1$. Since $\lambda_0^* = 0$, therefore (6) is equivalent to the input-based SE-BCC model. Thus the input-based SE-BCC model is feasible. This completes the proof of *only if* part.

To establish the *if* part, we note that if the input-based SE-BCC model is feasible, then $\rho^* > 1$ is the maximum radial increase of all inputs preserving the efficiency of DMU_0. Therefore χ can not be bigger than ρ^*, otherwise DMU_0 will be inefficient under the original input-based BCC model. Thus the input-based SE-BCC model is infeasible. •

Theorem 7

The input-based SE-BCC model is infeasible if and only if $g^* < 1$, *where* g^* *is the optimal value to (7).*

$$\begin{aligned} g^* &= \max g \\ \text{s.t.} \sum_{\substack{j=1 \\ j \neq 0}}^{n} \lambda_j y_j &\geq g y_0; \end{aligned}$$

$$\begin{aligned}\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j &= 1 \\ \lambda_j &\geq 0, \quad j \neq 0.\end{aligned} \tag{7}$$

Proof

We note that for any λ_j $(j \neq 0)$ with $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j = 1$, the constraint $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j x_j \leq \rho x_0$ always holds. Thus the input-based SE-BCC model is infeasible if and only if $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j y_j \geq y_0$ does not hold for any λ_j $(j \neq 0)$ with $\sum_{\substack{j=1\\j\neq 0}}^{n}$. This means that the optimal value to (7) is less than one, *i.e.*, $g^* < 1$. •

3.5 Infeasibility and Returns to Scale

Lemma 4:

If DMU_0 exhibits IRS, then the input-based SE-BCC model is feasible and moreover $\rho^* > 1$, where ρ^* is the optimal value to the input-based SE-BCC.

Proof

Let $\vartheta = 1/\rho$, then the input-based SE-BCC model becomes

$$\begin{aligned} &\max \vartheta \\ \text{s.t.} \sum_{\substack{j=1\\j\neq 0}}^{n} \hat{\lambda}_j x_j &\leq x_0; \\ \sum_{\substack{j=1\\j\neq 0}}^{n} \hat{\lambda}_j y_j &\geq \vartheta y_0; \\ \sum_{\substack{j=1\\j\neq 0}}^{n} \hat{\lambda}_j = \vartheta &= \frac{1}{\rho}; \\ \rho,\ \vartheta,\ \hat{\lambda}_j &\geq 0. \end{aligned} \tag{8}$$

where $\hat{\lambda}_j = \vartheta\lambda + j$ $(j \neq 0)$.

Since DMU_0 exhibits IRS, then by Lemma 1, $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j^* < 1$ in all optima to the following output-based SE-CCR model:

$$\begin{aligned} &\max \varphi \\ \text{s.t.} \sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j x_j &\leq x_0; \\ \sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j y_j &\geq \varphi y_0; \\ \varphi,\ \lambda_j &\geq 0. \end{aligned} \tag{9}$$

Let $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j^* = \vartheta < 1$. Since DMU_0 is CCR inefficient, therefore $\varphi > 1$ and hence $\varphi > \vartheta$ is a feasible solution to (9). This in turn indicates that ϑ and λ_j^* $(j \neq 0)$ with $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j^* = \vartheta$ is a feasible solution to (8). Therefore the input-based SE-BCC model is feasible. Furthermore, by Proposition 2, we have that $\rho^* > 1$, where ρ^* is the optimal value to the input-based SE-BCC model. •

Theorem 8

If the input-based SE-BCC model is infeasible, then DMU_0 exhibits DRS or CRS.

Proof

If DMU_0 exhibits IRS, then by Lemma 4, the input-based SE-BCC model is feasible, which leads to a contradiction. •

Theorems 6 and 7 indicate that if the input-based SE-BCC model is infeasible, then DMU_0 is one of the *endpoints*. Furthermore if DRS prevail, then DMU_0 is an *right endpoint* (see Figure 1).

3.6 Other Input-based Super-efficiency Models

Now consider the input-based SE-NIRS and SE-NDRS models.

Theorem 9

The input-based SE-NDRS model is always feasible.

Proof

Since $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j \geq 1$ in the input-based SE-NDRS model, therefore there must exist some $\tilde{\lambda}_j$ with $\sum_{\substack{j=1\\j\neq 0}}^{n} \tilde{\lambda}_j > 1$ such that $\sum_{\substack{j=1\\j\neq 0}}^{n} \tilde{\lambda}_j y_j \geq y_0$ holds. Note that $\sum_{\substack{j=1\\j\neq 0}}^{n} \tilde{\lambda}_j x_j \leq \rho x_0$ can always be satisfied by a proper ρ^3. Thus the input-based SE-NDRS model is always feasible. •

Lemma 5

The input-based SE-NIRS model is infeasible if and only if *the input-based SE-BCC model is infeasible.*

Proof

The *only if* part is obvious and hence is omitted. To establish the *if* part, we suppose that the input-based SE-NIRS model is feasible. *I.e.*, we have a feasible solution with $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j \leq 1$ for the input-based SE-NIRS model. If $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j = 1$, then this solution is also feasible for the output-based SE-BCC model. If $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j < 1$, let $\sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j = e < 1$. Then $\sum_{\substack{j=1\\j\neq 0}}^{n} \hat{\lambda}_j y_j \geq \sum_{\substack{j=1\\j\neq 0}}^{n} \lambda_j y_j \geq y_0$, where $\hat{\lambda}_j = \lambda_j / e$ $(j \neq 0)$ and $\sum_{\substack{j=1\\j\neq 0}}^{n} \hat{\lambda}_j = 1$. Therefore $\hat{\lambda}_j$ $(j \neq 0)$ is a feasible solution to the output-based SE-BCC model. Both possible cases lead to a contradiction. Thus the output-based SE-NIRS model is infeasible if the output-based SE-BCC model is infeasible. •

On the basis of this lemma, we obtain

Theorem 10

For a specific extreme efficient $DMU_0 = (x_0, y_0)$, we have:

(a) The input-based SE-NIRS model is infeasible if and only if *$(\chi x_0, y_0)$ is efficient under the original BCC model for any $1 \leq \chi < +\infty$;*
(b) The input-based SE-NIRS model is feasible if and only if *$g^* < 1$, where g^* is the optimal value to (7)*

Remark

If $DMU_0 \in E$ under the NIRS model, then DMU_0 exhibits DRS or CRS. By Proposition 1, the DMU_0 also lies on the BCC frontiers that satisfy DRS or CRS. *I.e.*, the original

Super-efficiency DEA models		Feasibility	Returns to scale
Output-based	SE-BCC	Theorem 2 (model (2))	DRS
	SE-NIRS	always feasible	always feasible
	SE-NDRS	Lemma 3, Theorem 2	Corollary 1 (a)
Input-based	SE-BCC	Theorem 7 (model (7))	IRS
	SE-NIRS	Lemma 5, Theorem 7	always feasible
	SE-NDRS	always feasible	Corollary 1 (b)

Table 1: Super-efficiency DEA models and infeasibility

BCC and NIRS models are identical for DMU_0. Thus $(\chi x_0, y_0)$ is also efficient under the original NIRS model for any $1 \leq \chi < +\infty$.

Furthermore, Theorems 3 and 8 demonstrate that the possible infeasibility of the output-based and input-based SE-BCC models can only occur at those extreme efficient DMUs exhibiting IRS (or CRS) and DRS (or CRS) respectively. Note that IRS and DRS are not allowed in the NIRS and NDRS models respectively. Therefore we have the following corollary.

Corollary 1

(a) If $DMU_0 \in E$ exhibits DRS, then all output-based super-efficiency DEA models are feasible;

(b) If $DMU_0 \in E$ exhibits IRS, then all input-based super-efficiency DEA models are feasible.

Andersen and Petersen (1993) state that their procedure for ranking the efficient DMUs is applicable under the SE-BCC and SE-NIRS models. However, the results of this paper show that their statement is not totally correct. The ranking of the total set E of observations is impossible because of the infeasibility. However their procedure can be applied to the input-based SE-NDRS model and the output-based NIRS model.

Zhu (1996) employed the SE-CCR model to study the stability of the efficiency classifications under the CCR model. However he failed to notice that one can also employ the SE-BCC model to study the stability of the efficiency classifications under the BCC model or the additive model (see Seiford and Zhu, 1998b). By Theorems 1 and 6, we know that infeasibility indicates that the inputs of an extreme efficient DMU_0 can be proportionally increased without limit or that the outputs can be decreased in any positive proportion, while preserving the efficiency of DMU_0. This indicates that the efficiency of DMU_0 is always stable under the proportional data changes.

Cooper, Kumbhakar, Thrall and Yu (1995) provide an approach to determine the *endpoint* locations of the extreme efficient DMUs in the case of two-input and one-output (or two-output and one-input). Our method here generalizes their approach to the case of multiple inputs and outputs.

Finally, on the basis of the results developed here, one is able to choose the correct version of the super-efficiency DEA models (see Table 1). Models (2) and (7) are useful in the determination of infeasibility while Theorems 1 and 6 are useful in the sensitivity analysis of efficiency classifications.

DMU No.	RTS Classifications	Output-based SE-BCC	$\hbar^*$	Input-based SE-BCC	g^*
1	CRS	0.36130	0.06934	infeasibility	0.36130
2	DRS	0.98070	0.10133	1.03800	1.30267
6	CRS	0.76935	0.20801	1.40292	1.79939
8	CRS	0.22404	0.27397	4.86734	1.12961
21	CRS	0.91118	0.69235	1.16355	4.14149
23	CRS	0.80103	0.94235	1.12732	5.91508
24	CRS	0.90426	0.71772	1.13029	5.59984
25	CRS	infeasibility	1.13243	1.24068	8.53306
26	CRS	infeasibility	1.11127	1.42393	4.84178
27	IRS	infeasibility	2.30463	2.30463	44.54216

Table 2: Infeasibility of the SE-BCC.

4. CONCLUSIONS

The current paper studies the infeasibility of the super-efficiency DEA models which have been widely employed in determining stability of the efficiency classifications, measuring technology and productivity changes, ranking DEA efficient DMUs, identifying extreme efficient DMUs and solving two-person games. Necessary and sufficient conditions for infeasibility are provided. Our results indicate that the use of the super-efficiency DEA models should be restricted in some situations. However, the use of the super-efficiency DEA models in the sensitivity analysis of efficiency classifications can be generalized from the CCR model (Zhu, 1996 and Seiford and Zhu, 1998a) to the situation of non-constant returns to scale (Seiford and Zhu, 1998b). In addition, infeasibility does provide information on the endpoint positions of the extreme efficient DMUs. Hence, we obtain insight into the structure of the data domain.

Finally, we note that the super-efficiency BCC models could also be used to estimate RTS. This is a possible new usage of the super-efficiency DEA models. In our opinion the super-efficiency DEA models provide important managerial information and should be a standard part of the DEA analyst tool kit.

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Appendix

We illustrate the infeasibility of the SE-BCC model with a data set consisting of 28 Chinese cities (DMUs) from Charnes, Cooper and Li (1989). There are three outputs (gross industrial output value, profit & taxes, and retail sales) and three inputs (labor, working funds, and investment).

Table 2 provides the extreme efficient DMUs under the BCC model. DMU numbers correspond to the original ones in Charnes, Cooper and Li (1989)[4]. We use the RTS method suggested in Zhu and Shen (1995) which is independent of the possible multiple optimal lambda values in the CCR model to estimate the RTS on those DMUs. Columns 3 and 5, respectively, report the scores obtained from the output-based SE-BCC model and the input-based SE-BCC model. Columns 4 and 6 report the optimal values to (2) and (7) respectively.

[4]DMU8 was misclassified as inefficient in their study (see Ali and Seiford, 1993).

For example, DMU27, which exhibits IRS, is infeasible for the output-based SE-BCC model with $\hbar^* = 2.30463 > 1$. DMU1, which exhibits CRS, is infeasible for the input-based SE-BCC model with $g^* = 0.36130 < 1$.

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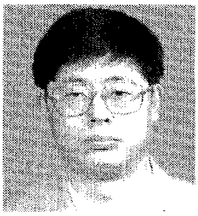

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