Measuring efficiency in DEA in the presence of common inputs

Sonia Valeria Avilés-Sacoto, Wade D. Cook, David Güemes-Castorena & Joe Zhu

To cite this article: Sonia Valeria Avilés-Sacoto, Wade D. Cook, David Güemes-Castorena & Joe Zhu (2019): Measuring efficiency in DEA in the presence of common inputs, Journal of the Operational Research Society, DOI: 10.1080/01605682.2019.1630329

To link to this article: https://doi.org/10.1080/01605682.2019.1630329

Published online: 27 Aug 2019.
Measuring efficiency in DEA in the presence of common inputs

Sonia Valeria Avilés-Sacoto\textsuperscript{a}, Wade D. Cook\textsuperscript{b}, David Güemes-Castorena\textsuperscript{c} and Joe Zhu\textsuperscript{d}

\textsuperscript{a}Industrial Engineering Department, Universidad San Francisco de Quito USFQ, Campus Cumbayá, Quito, Pichincha, Ecuador; \textsuperscript{b}Schulich School of Business, York University, Toronto, Ontario, Canada; \textsuperscript{c}Tecnologico de Monterrey, Campus Monterrey, N.L., Monterrey, México; \textsuperscript{d}School of Business, Worcester Polytechnic Institute, Worcester, MA, USA

\section*{ABSTRACT}
Data envelopment analysis (DEA) is a methodology for evaluating the relative efficiencies of a set of decision-making units (DMUs). It is commonly assumed that the DMUs are independent of one another, in that each has its own quantities of a set of inputs and outputs. In case this assumption of independence of DMUs holds, decreasing the inputs of one DMU will not affect the inputs of others. The current paper moves beyond the conventional framework and examines a problem setting where there is an interdependence among the DMUs. Consider the case where the members of a given subgroup of DMUs have an input in common, such as would be the case if a set of highway maintenance crews in a district are under the jurisdiction of a district supervisor and district-level resources. The efficiency measurement difficulty created by this “shared” resource phenomenon is that in attempting to move an inefficient crew towards the frontier by reducing that shared resource (hence penalising that crew), the other crews in that same district will be equally penalised. Specifically, decreasing district resources in relation to their impact on a maintenance crew will cause that resource to decrease as well for other members of the same group. The conventional (input-oriented) DEA model that does not cater for such interdependence situations will fail to address this important issue. To capture this interdependence, we develop a new DEA-like methodology. One of the properties of this new methodology is that its production possibility set cannot be defined in the same manner as in the conventional DEA setting. This is due to the fact that when the DMU under evaluation is projected towards the frontier, the input/output structures of the other units in the same group are altered, unlike the conventional situation where the structures of the other DMUs remain fixed. We apply this new methodology to the problem of evaluating a set of departments in a university setting, where the departments are grouped under various faculties.

\section*{1. Introduction}

Data envelopment analysis (DEA) developed by Charnes, Cooper, and Rhodes (1978) is an optimisation tool for measuring the efficiency of each member of a set of decision-making units (DMUs), relative the other DMUs in that set. The conventional model assumes that each DMU possess its own quantities of a set of inputs \( \{x_i\}_{i=1}^{I} \) that generate or result in its own set of outputs \( \{y_{ij}\}_{j=1}^{J} \). Furthermore, it is assumed that the input and output bundles for any given DMU are in no particular way connected to those of the other DMUs. In more precise terms, the DMUs are assumed to be independent of one another.

In some situations, however, one of which we address herein, the independence assumption may be violated. Specifically, some form of interdependence connects the DMUs. Consider, for example, the situation wherein we wish to evaluate the efficiency of each of a set of automobile dealerships located across the country. The input bundle, unique to any given dealership, might include total numbers of vehicles available for sale, size of the maintenance department, numbers of salespersons, and local advertising. The output bundle would include number of vehicles sold per month, total revenue generated by the sales staff, and total revenue generated by vehicle repair shop staff. What is missing from the input profile is what we might term “block ads”. Specifically, sales numbers of any particular brand of automobile, such as a Subaru, are very much influenced by TV, radio, magazine, and newspaper ads. These ads are a form of “common” or shared input, benefiting all dealerships in the vicinity where those ads appear. Furthermore, the ads can vary from one vicinity to another. This means that the set of all dealerships can be organised into groups, whereby each dealership in a given group has its own dealer-specific inputs, and as well benefits from the common set of ads pertinent to that group of DMUs.

Another example where this common input concept appears, and one that we examine below, involves the measurement of efficiencies of
university departments or careers, with each department falling under the jurisdiction of a school or faculty. (Hereafter, we adopt the terms faculty and department.) Each department in each faculty has its own unique set of inputs and outputs, but at the same time all departments in a faculty generally benefit from a common resource provided by the faculty. A given faculty will commonly employ both academic (professorial staff) and non-academic staff (normally administrative employees, secretaries, etc.). In the case of professors, they are normally employed as dedicated staff attached to a given department, although we acknowledge that there can be "cross-appointments". In the case of administrative staff and secretaries, one often finds that they provide service to the full set of departments within a faculty. Thus, for all practical purposes academic staff can be considered as a department-level resource or input, while non-academic staff is faculty-level inputs.

In the following sections, we develop a DEA-based methodology that can be used to measure the efficiencies of a set of DMUs in the presence of shared or common resources, thus giving rise to a dependence situation among those DMUs.

Section 2 describes the university setting and the variables that reasonably characterise the operation of the various departments. With this setting as a backdrop, Section 3 reviews some of the relevant literature relating to the issue of measuring efficiency of university departments, as well as literature involving shared inputs and/or outputs. Section 4 develops a DEA type of methodology to evaluate the performance of a set of DMUs, where those DMUs form natural groupings that arise from the presence of shared resources. Section 5 applies the proposed methodology to a set of university departments. Conclusions and recommendations are detailed in Section 6.

### 2. Efficiency measurement in university departments

In this paper, we examine the efficiency of each member of a set of academic departments in a university setting. The university under study there were 42 academic departments, grouped into 10 faculties. For purposes of the study herein, and due to the privacy of the information provided, we have labelled the academic departments “AD1” to “AD42”, rather than using actual names. The academic departments play the roles of DMUs and are listed randomly in this study. The data used in the study are numerical; for the independent inputs, we used the department operating budget and total academic staff in each academic department, while research grants, total numbers of under-graduate degrees awarded, and the total numbers of research publications represent the outputs. Total numbers of non-academic staff are a type of shared input in that they provide service to the different academic departments which belong to a faculty.

Specifically, we define inputs and outputs as

- **Inputs**
  - Operating budget: This variable includes (1) the salaries and wages for academics and non-academic staff, (2) employee benefits, (3) services, supplies, travel, and other expenses, (4) maintenance, equipment, and utilities, and (5) transfers to endowment principal, and others. For public universities, federal and/or state support is included within the operating budget.
  - Total academic staff: This variable takes into account all professors, in a full time equivalent (FTEs) sense, within an academic department (full, associate, and adjunct professors). This variable is usually considered an input to the system.
  - Total non-academic staff: This variable appears under various names in different settings – Professional Staff, Administrative Staff, or Executive Staff (Sebalj, Holbrook, and Bourke, 2012). These are employees whose responsibility it is to assist professors and students, be involved in administrative duties involving teaching activities and research projects, and assist in managing academic activities. This is usually considered an input to the system, at the faculty rather than department level.

- **Outputs**
  - Research grants: Since academic staff do research, the amount of money that comes from internal and/or external grants awarded to professors in a department, is an important indicator of research activity in that department (Abd Aziz, Mohd Janor, & Mahadi, 2013).
  - Total numbers of under-graduate degrees awarded: This variable is usually related to the quality and quantity of teaching by academic staff (Abd Aziz et al., 2013). We point out that we are assuming that each graduating student has a home department, even though he/she will normally take courses outside that department. So, the student majoring in Industrial Engineering, for example, will also take some courses in the social sciences. It is also the case that some students may do a double major, meaning that he/she may arguably belong to more than one faculty.
  - Weighted total publications: This variable is intended to be an aggregate of the total number of papers in journals, research books published, book chapters published, and papers presented at academic conferences. For the study herein, we have imposed a weight to each of the different types of
publications. This is intended as a key indicator for research activity of a department. We do acknowledge that it is common to find disagreement among the opinions of people about the quality or importance of research works. Some tend to emphasise the publication of papers in journals more than books. For the study herein, we have computed a weighted publication by: Weighted publications = 1*(Number of papers in journals) + 0.5*(Number of papers in conference proceedings) + 0.5*(Number of book chapters) + 2*(Number of research books).

The current research fits within a large volume of efficiency studies in education, we review some of this literature below. What is unique about the particular setting examined herein is the presence of factors that create a form of interdependence among the DMUs, as discussed in Section 1. This interdependence in the current setting appears in the form of a shared resource (administrative staff.) The general concept of shared resources is, of course, not new in the DEA literature, and we therefore review below, as well, some of the relevant literature. We specifically point to key differences between the current work and previous studies.

3. Literature review

3.1. Literature relating to efficiency in academic departments

In an economy, universities play a vital role as the place where people gain knowledge through the process of teaching-learning through professors and students. Universities can be defined as “... major concentrations of political, social, economic, intellectual and communicative resources” (Marginson, 2014). Although several characteristics make the difference between one university and another, such as the governance, funding, and organisation, the common characteristic that universities generally share is the structure (Smart, 2002). The traditional structure of a university is one where different academic departments or careers, such as Industrial Engineering, Mechanical Engineering, etc., are commonly grouped together as a school or faculty (e.g., Engineering), according to the knowledge-based involved and research interests.

In the past several years, measurement of the performance of the different academic departments has gained importance. Some of the earliest studies of academic institutions were those due to Bessent, Bessent, Charnes, and Cooper (1983) and Johnes and Johnes (1993), involving settings where technical efficiency focuses on inputs such as teaching staff and research funding, and outputs are measured in terms of courses taught and students supervised. Another study was that due to Beasley (1995) where a model is developed for measuring efficiencies of academic departments in terms of both teaching and research. Some studies include the measurement of efficiency in the terms of how well economic resources are used (Agha, Kuhail, Abdelnabi, Salem, & Ghanim, 2011; Kao & Hung, 2008); the level of education in terms of teaching, student placements and research (Abd Aziz et al., 2013; Tyagi, Prasad Yadav, & Singh, 2009); the level of research (Agasisti, Catalano, Landoni, & Verganti, 2012); and staff performance (Sirbu, Cimpoies & Racul, 2016). In those studies, commonly used variables include academic staff, administrative budgets, publications, and research grants. Avilés-Sacoto, Cook, & Güemes-Castrorena (2014) and Avilés-Sacoto, Cook, Güemes-Castrorena, and Cantú-Delgado (2015) look at two-stage and “time staged” models for measuring the competition among higher education institutions.

Table 1 details recent studies in DEA that have used these variables at evaluating academic departments.

3.2. Shared input and output studies

Most DEA studies presume that the members of the set of DMUs to be evaluated, use their own inputs to produce their own outputs, and that they generally operate independently of one another. Therefore, efficiency scores and accompanying projections to the best practice frontier should be a realistic portrayal of the efficiency standing of each DMU. There are cases, however, where the DMUs under evaluation are interdependent due to some form of sharing of inputs or outputs within any DMU. In such situations, the efficiency and the projections can be affected. As mentioned above, Beasley (1995) was one of the first authors to examine efficiency in academic departments which involved sharing inputs across the two functions, research and teaching. He presented nonlinear models for splitting inputs between the teaching and research roles. Cook, Hababou, and Tuenter (2000) and, Cook and Hababou (2001) extended Beasley’s ideas by developing linear models for resource sharing and splitting across different components or functions within bank branches. Tsai and Molinero (2002) used a weighted objective function using the VRS (Variable Returns to Scale) model of Banker, Charnes, and Cooper (1984) to examine multiple component situations in the UK health service. The analyses of shared inputs and share outputs are usually applied in bank branches such as the one from Amirteimoori and Nashtaei (2006) and that from Cook and Zhu (2011).
In some recent literature considering shared inputs, there appear to be two major settings, namely, parallel and serial settings (see Ding, Dong, Liang, & Zhu, 2017). In both cases, it appears that the shared input is split and assigned to the relevant components (either the parallel components or divisions, or the serial stages constituting the relevant parts (components) of the DMU). One study that involves sharing inputs or outputs is the study of Chen, Du, Sherman, and Zhu (2010) that evaluates a set of DMUs that share inputs to produce multiple outputs, in the context of a two-stage process. Among the recent studies done in DEA considering shared inputs is that due to Wu, Zhu, Ji, Chu, and Liang (2016) in which the DMUs are regions in China. Therein, the process of sharing inputs takes into account a two-stage situation, where the shared resources are used in both the first and second stages. Along the same line, the study done by Jianfeng (2015) evaluates the presence of shared inputs in two-stage settings. That work considers also the situation where outputs from the first stage are inputs of the second stage and also there are free intermediate inputs. Also, the study done by Bian, Hu, and Xu (2015) develops a DEA methodology to evaluate DMUs that have parallel sub-units with shared inputs and outputs. The research therein is related to the earlier research by Cook and Hababou (2001), where one can think of the sales and service components in bank branches as parallel sub-processes and the work on parallel processes by Kao (2009). In the Bian et al. (2015) research looks at both shared inputs and shared outputs. The proposed approach is used to measure 18 railway firms’ performances.

A strain of research referred to as “multi-activity” data envelopment analysis (MDEA). The idea of MDEA was originally proposed by Beasley (1995) as discussed above. Along a similar line, Chen et al. (2013) treated the concept of shared inputs in a study of a set of farmer’s cooperatives which have several parallel business activities. The framework of the study provides the possibility to get efficiency scores based on a comparison of individual activities among peers and then embedding them into a maximisation of the overall achievement with constraints on shared inputs.
In the study by Ding et al. (2017), referred to above, the concept of “goal congruency” is examined. There, the shared resources allocation issue is handled in a bank branch setting. The idea of goal congruency sets out to evaluate an entire DMU and its divisions under certain conditions. The shared resources make it possible to improve a plan for goal congruency.

A somewhat related line of research involves a different form of sharing in the sense that certain variables can play both input and output roles. Early studies by Beasley (1990) and Beasley (1995) looked at the role of research funding in a university setting in that it serves as both an input and output. This “dual role” idea has been further studied in Cook, Green, and Zhu (2006).

Another line of somewhat related research is that involving non-homogeneous DMUs in DEA (see Cook, Imanirad, Harrison, Rouse, & Zhu, 2013 and Imanirad, Cook, & Zhu, 2013). These studies look at shared resources and the splitting of these across different output groups within each DMU.

An area of research related to that presented herein is that involving the allocation of a fixed cost or resource across a set of DMUs. This idea was first considered in Cook and Kress (1999) and later examined in Cook and Zhu (2005) and Cook, Du, Liang, and Zhu (2014). The basic idea in these earlier works is to develop a DEA score prior to considering the fixed factor and then allocating (or splitting) that factor after the fact. Our work herein takes account of shared resources while deriving efficiency scores, but not splitting that resource across the DMUs involved.

The majority of the problem settings discussed above generally involve splitting up inputs and/or outputs shared among a set of components or functions within a DMU and allocating the resulting proportions among those components or functions: Splitting resources in each member of a set of bank branches and allocating those split amounts to sales and service functions in that branch; splitting up resources within a university and allocating them to the teaching and research functions within that university, etc. In the current paper, however, we consider situations that are distinctly different than those settings discussed in the previous literature. Specifically, we look at settings where DMUs are organised into groups, with each group sharing an input, which cannot be split among the DMU members therein. That being the case, the conventional DEA model fails to provide a proper framework within which to evaluate efficiency in the presence of the type of sharing referred to herein. In the following section, we present a methodology to handle such situations.

4. Methodology

Consider the situation wherein each of \(n\) DMUs, \(j = 1, \ldots, n\) is to be evaluated in terms of \(I\) inputs \(x_{ij}, i = 1, \ldots, I\), and \(R\) outputs \(y_{rj}, r = 1, \ldots, R\). The conventional input-oriented DEA model of Charnes, Cooper, and Rhodes (1978), in envelopment form for a given DMU \(j_0\), is given by model (4.1) and illustrated by Figure 1. We have included the slack variables in the inequality constraints allowing one to easily differentiate between weak and strong efficiency. This is discussed later in the paper. Specifically, it is desirable to have radial projections (towards the origin) hit the frontier proper such as for the case of DMU F. In case the projection fails to make it to the actual frontier, but rather hits an extension of the frontier, such as for DMU H, then the slack variable represents a gap between the projected point and a legitimate point on the frontier proper.

\[
\begin{align*}
\text{Min} & \quad \theta & \quad \text{(4.1a)} \\
\text{subject to} & \quad \sum_{j=1}^{n} \lambda_{ij} x_{ij} + s_{i}^{-} = \theta x_{i0}, \forall i & \quad \text{(4.1b)} \\
& \quad \sum_{j=1}^{n} \lambda_{rj} y_{rj} - s_{r}^{+} = y_{r0}, \forall r & \quad \text{(4.1c)} \\
& \quad \lambda_{ij}, s_{i}^{-}, s_{r}^{+} \geq 0, \theta \text{ unrestricted in sign.} & \quad \text{(4.1d)}
\end{align*}
\]

In the conventional DEA situation, the production possibility set \(P\) is characterised by the following properties:

A1: The observed DMUs belong to \(P\).
A2: Any positive scalar multiple of a member of \(P\) is also a member of \(P\).
A3: For an activity \((x,y)\) in \(P\), any semipositive activity \((\tilde{x},\tilde{y})\) with \(\tilde{x} \geq x\) and \(\tilde{y} \leq y\) is also in \(P\).
A4: Any semipositive combination of activities in \(P\) is also in \(P\).
Given these properties, the production possibility set is defined by

\[ P = \left\{ (\bar{x}, \bar{y}) | \bar{x}_i \geq \sum_{j=1}^{n} \lambda_j x_{ij}, \bar{y}_j \leq \sum_{j=1}^{n} \lambda_j y_{ij}, \lambda_j \geq 0 \right\}. \]

Model (4.1) presumes that all DMUs act independently of one another, meaning that for any given DMU (department) \( j_0 \) we can reduce input consumption \( x_{ij} \) by the factor \( \theta \) in each of the \( I \) dimensions, and that this reduction involving this particular DMU in no way impinges upon the other DMUs present. One might take the line of argument that even if there is a group-level discretionary input \( x_{ij}^k \) held by group \( K_k \), it can, in some instances, be split or shared in some manner among the departments \( j_k \) within faculty \( K_k \) in amounts \( x_{ij}^{k+1} \). If this is done, then for any given department \( j_k \) within \( K_k \), a reduction in \( x_{ij}^{k+1} \) could presumably occur with no impact on the other members of \( K_k \), meaning that model (4.1) could be applied to arrive at a DEA score for that department. In other words, there can be problem settings where, for example, we have two departments \( D_1 \) and \( D_2 \) in a faculty with three faculty secretaries \( S_1, S_2, \) and \( S_3 \), and where we might have secretaries \( S_1 \) and \( S_2 \) specifically allotted to \( D_1 \), and the third secretary \( S_3 \) allotted to \( D_2 \). If we now suppose that \( D_1 \) is inefficient, then the faculty could consider reducing the secretary complement for that department by moving \( S_1 \) or \( S_2 \) (or both) to a part time status (assuming union regulations do not come into effect). The point we are making here is that by splitting the secretarial compliment, we would be able to remove the interdependence among the DMUs (departments), thereby permitting each department to operate autonomously.

The flaw in the above reasoning for some situations, and for the particular situation at hand, however, is that the type of shared input under consideration herein is one that cannot reasonably be split in the manner described. Specifically, we assume that the secretarial pool is not split up across the departments, meaning that each and every secretary is available to all departments. That being the case, any reduction in administrative staff in a faculty (e.g., a reduction in the number of secretaries from 3 to 2) is assumed to impact all departments equally, regardless of which are efficient and which are not. A similar situation would be that involving block ads for automobiles, as described earlier. Reducing the number of TV ads in a particular region would have the same negative effect on all dealerships in that region, regardless of the efficiency status of any given dealership. Given such settings, the conventional model fails to capture the dynamics of shared inputs in that model (4.1) presumes that an adjustment \( \theta x_{ij}^{k+1} \) in the shared input when evaluating the efficiency status of one DMU \( j_0 \in K_k \), does not influence the status of other DMUs \( j \in K_k \).

To formalise these ideas, let the inputs specific to department \( j \) be denoted as \( x_{ij}, i = 1, \ldots, I \), and let \( x_{ij}^{k+1} \) denote the faculty-level input held by each department \( j \) in faculty \( k \). Let us adopt the notation \( K_1, K_2, \ldots, K_L \) for the \( L \) faculties under consideration. For example, for faculty \( 1 \), \( K_1 \) consists of departments 1, 2, 3, and 4 (see Table 2). The proper representation of the dynamics of the shared input is to view the input adjustment \( \theta x_{ij}^{k+1} \) for any given \( j_0 \in K_k \), as being the same (i.e., \( \theta x_{ij}^{k+1} \)) for every other \( j \) in that same faculty. (For convenience we have adopted the notation \( x_{ij}^{k+1} \) even though the compressed notation \( x_{ij}^{k+1} \) would suffice, given that each DMU \( j \) in \( K_k \) has access to the same shared resource.) Now consider the following model (4.2) for a DMU \( j_0 \in K_k \).

\[
\begin{align*}
\text{Min } \theta & \quad \text{subject to} \\
\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- - \theta x_{ij}^- & = 0, i = 1, \ldots, I, \quad \text{(4.2a)} \\
\sum_{j \in K_k} \lambda_j x_{ij}^f + \lambda_j x_{ij}^{k+1} + \sum_{j \in K_k} \lambda_j \theta x_{ij}^{k+1} + s_i^- & = \theta x_{ij}^{k+1}, \quad \text{(4.2b)} \\
\sum_{j=1}^{n} \lambda_j y_{jr} - s_r^+ & = y_{jr}, r = 1, \ldots, R, \quad \text{(4.2d)} \\
\lambda_j, s_i^-, s_i^+, s_r^+ & \geq 0, \theta \text{ unrestricted in sign.} \quad \text{(4.2f)}
\end{align*}
\]

Note that constraint (4.2c) replaces \( \lambda_j \) by \( \lambda_j \theta \) for all departments (DMUs) \( j \) in \( K_1-j_0 \), thereby creating a nonlinearity. This expression accounts for the fact that as inputs for a DMU \( j_0 \in K_k \) are reduced, so the other members are \( j \in K_k \). For all other DMUs, \( j \) only the multiplier \( \lambda_j \) is applied. One might question the second term on the left side of (4.2c), namely, \( \lambda_j x_{ij}^{k+1} \), rather than \( \lambda_j \theta x_{ij}^{k+1} \). The argument for doing this is the same as that in the conventional model, where on the left side of constraints (4.1b) we write the term \( \lambda_j x_{ij} \), rather than \( \lambda_j \theta x_{ij} \). The point is that we wish to evaluate the DMU \( j_0 \) against its efficient peers. If \( j_0 \) is not efficient, then the evaluation will be against only the efficient peers, and DMU \( j_0 \) (with its inefficient status) could be excluded altogether from the left side of the inequality. If \( j_0 \) is efficient, then \( \theta = 1 \), and it does not matter whether we use the expression \( \lambda_j x_{ij} \) or \( \lambda_j \theta x_{ij} \).
In the conventional CCR model (4.1), as reflected in Figure 1, we note that any projected value $\theta x_{hjo}$ of a given input $i$ for DMU $j_o$ cannot be less than $\min_{i=1,...,n} \{ x_{hj} \}$, hence the projection must stay within the frontier and its extensions. Thus, the projection is guaranteed not to push through the frontier. In the conventional DEA situation, the data characterising any given DMU remains unaltered in the optimisation process of (4.1). In other words, the input data $x_{hj}$ on the left side of (4.1b) and output data $y_{hj}$ on the left side of (4.1c) remain fixed. In the situation, however, where a shared input is present, this is not the case. Specifically, when deriving the efficiency of any given DMU $j_o$ in group $K_k$, constraint (4.2c) acts to alter the shared-input data for the remaining DMUs $j$ in that set ($K_k$). Specifically, when evaluating a DMU $j_o$ in $K_k$, $x_{hj}$ is replaced by $\theta x_{hj}^{k}$ for all $j \in K_k - j_o$. Given this fact, and given that we wish to insure that projections do not pass through the frontier, we impose constraint (4.2e). This being the case, the form of the production possibility set is less straightforward or more accurately is a function of $j_o$ and $\theta$. More to the point, one can specify for each DMU $j_o$ in group $K_k$ a type of parametric production possibility set $P_{k,hj_o}(\theta)$, which is given by

$$P_{k,hj_o}(\theta) = \left\{ (x,y) : \sum_{i=1}^{n} \lambda_{i} x_{hij} \leq x_{hj}, i = 1, \ldots, I, \right.$$  

$$\left. \sum_{j \in K_k} \lambda_{j} x_{hj+1j} \leq x_{hj+1j}, \right.$$  

$$\left. \sum_{j \in K_k} \lambda_{j} x_{hj}^{k} \geq x_{hj}^{k}, \right.$$  

$$\left. \sum_{j \in K_k} \lambda_{j} y_{hj} \geq y_{hj}, r = 1, \ldots, R, \lambda_{j} \geq 0, \theta \right\}.$$  

An illustration

To illustrate model (4.2) consider the simple situation involving six DMUs or departments. The first
three faculties have one department each, labelled A, B, and C. The fourth faculty has three departments labelled D, E, and F. The data are presented in Table 3.

Here, there is a single output “O” with each department having one unit of that output. Let us assume that there are two inputs, \( x_1 \), \( x_2 \), with the former representing a ‘regular’ input say total academic staff, and the latter, a shared input representing non-academic staff. Figure 2 shows the inputs in a two-dimensional space; the output can be ignored. Let us consider two situations.

Situation 1: Here we treat model (4.2) directly as stated. In this case all DMUs are efficient, with B, C, and D being strongly efficient and A, E, and F being weakly efficient. Figure 2 displays this situation.

Situation 2: If we ignore constraint (4.2e), then DMUs E and F are deemed inefficient. In that situation Department E has an efficiency score of 0.80 and in a radial sense is projected to point \( E' \) with coordinates (20, 6.4). DMU F is projected radially to point \( F' \) with coordinates (20, 5.16) and has an efficiency score of 0.645 (Figure 3).

Solving model (4.2)

In the following section, we apply model (4.2) to evaluate the efficiencies of academic departments. There would appear to be two approaches to solving (4.2). The first approach is to treat the model as a general non-linear programming problem, using a non-linear software routine such as GRG within Solver. This approach, however, has the disadvantage that solutions found are locally, but not necessarily globally optimal. The second approach is to treat \( \theta \) as a parameter rather than directly as a variable. For any set value of the parameter, model (4.2) is a linear programming problem. This latter approach is used in the following section.

5. Evaluating academic departments within a university

Table 2 displays data on 42 academic departments at a university. Specifically, the table provides the inputs in terms of operating budget in US dollars, total academic staff and total non-academic staff, an input that is shared among the different DMUs that fall into a group, in this case the faculty. Outputs take the form of total research funds in US dollars, total numbers of undergraduate students graduated in an specific year, and total weighted publications. The latter is described above. As mentioned previously, we have codified each of the academic departments with the prefix “AD” and a number (1 … 42).

If we apply the conventional model (4.1), efficiency scores would appear as shown in column 3 of Table 4. We have not bothered to include the slack variables for this case. We point out that in applying model (4.1), which assumes that DMUs operate independently of one another, we have split the shared input (faculty-level non-academic staff) equally across the departments in each faculty. Shown in columns 4-14 are the multipliers \( \lambda_j \) arising from the solution for each of the DMUs. Eleven of the DMUs are declared efficient (DMUs 1, 5, 11, 13, 16, 21, 23, 32, 39, 41, and 42).

Applying the new model (4.2), the results are as presented in Table 5. The essential difference between the application of this model and the conventional model is that the latter can have positive \( \lambda \) values corresponding to efficient units only.

![Figure 2. Efficient frontier for model (4.2).](image)

![Figure 3. Efficiency with constraint (4.2e) omitted.](image)
Table 4. Results from conventional model (4.1).

<table>
<thead>
<tr>
<th>Group</th>
<th>DMUs</th>
<th>Value objective function</th>
<th>Lambda 1</th>
<th>Lambda 5</th>
<th>Lambda 11</th>
<th>Lambda 13</th>
<th>Lambda 16</th>
<th>Lambda 21</th>
<th>Lambda 23</th>
<th>Lambda 32</th>
<th>Lambda 39</th>
<th>Lambda 41</th>
<th>Lambda 42</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td>0.429</td>
<td>0.036</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.429</td>
<td>0.036</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.608</td>
<td>0.354</td>
<td>0.418</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.247</td>
<td>0.023</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.435</td>
<td>0.023</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.452</td>
<td>0.057</td>
<td>0.259</td>
<td>0.160</td>
<td>0.106</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>0.226</td>
<td>0.079</td>
<td>0.429</td>
<td>0.036</td>
<td>0.418</td>
<td>0.076</td>
<td>0.608</td>
<td>0.354</td>
<td>0.259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>0.729</td>
<td>0.298</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.263</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 5. Results from applying the new methodology model (4.2).

<table>
<thead>
<tr>
<th>Group DMUs</th>
<th>Parametric Lambda 1</th>
<th>Lambda 2</th>
<th>Lambda 5</th>
<th>Lambda 10</th>
<th>Lambda 11</th>
<th>Lambda 12</th>
<th>Lambda 13</th>
<th>Lambda 16</th>
<th>Lambda 21</th>
<th>Lambda 23</th>
<th>Lambda 25</th>
<th>Lambda 32</th>
<th>Lambda 39</th>
<th>Lambda 42</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>SR1</th>
<th>SR2</th>
<th>SR3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.083</td>
<td>0.175</td>
<td>0.089</td>
<td>1.933</td>
<td>4.009</td>
<td>0.61</td>
<td>0.081</td>
<td>0.312</td>
<td>0.165</td>
<td>0.004</td>
<td>5.348</td>
<td>13.491</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0.102</td>
<td>0.044</td>
<td>0.047</td>
<td>126.547</td>
<td>13.798</td>
<td>204.463</td>
<td>5.852</td>
<td>10.922</td>
<td>6.132</td>
<td>0.005</td>
<td>3.631</td>
<td>11.438</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>0.5</td>
<td>0.006</td>
<td>0.141</td>
<td>0.841</td>
<td>3.121</td>
<td>20.03</td>
<td>8.311</td>
<td>0.005</td>
<td>10.31</td>
<td>5.231</td>
<td>0.005</td>
<td>4.121</td>
<td>13.031</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>0.31</td>
<td>0.078</td>
<td>0.002</td>
<td>0.091</td>
<td>0.141</td>
<td>1.121</td>
<td>0.141</td>
<td>0.005</td>
<td>0.005</td>
<td>1.121</td>
<td>0.005</td>
<td>1.121</td>
<td>0.141</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The values in the table represent the results obtained from applying the new methodology model. The columns indicate the different parameters and results for each group DMUs.
Specifically, in model (4.1), each inefficient DMU is evaluated against those DMUs that reside on the efficient frontier. In the case of model (4.2), however, this is not necessarily true. That is, certain inefficient DMUs can, in an altered form, act as a peer for other inefficient units. This is the case for DMU 12. Consider the results for the inefficient DMU 14 that has a score of 0.35. We observe that this DMU is evaluated against DMUs 12, 16, and 32. That is, \( \lambda_{12} = .714 \), \( \lambda_{16} = .256 \), \( \lambda_{32} = .005 \), and all other lambda variables are zero. However, DMU 12 is not on the frontier, hence is inefficient as its DEA efficiency score is 0.6. In the conventional DEA model (4.1), all DMUs are fixed, while in (4.2) when evaluating a DMU \( f_0 \in K_k \), all other members \( j \in K_k \) appear in altered format \( \theta x_{1,1j} \) (hence are not fixed). What is happening here as that there are two types of DMUs, namely, those that remain fixed (and are not altered in the optimisation process) and those that are altered during the optimisation process. In model (4.1) only fixed DMUs are present, while (4.2) permits both types. Since DMUs 12 and 14 are in the same faculty group, any percentage reduction of the shared input for DMU 14 means that the same percentage reduction will apply as well to DMU 12. Since the efficiency score for DMU 12 is larger than that for DMU 14, the former will be closer to the frontier as defined by \( \theta \) than is the case for the latter. So the important point to be made here is that in evaluating DMU 14 it is not the shared input \( x_{1,1j} \) for DMU 12 that is a peer for DMU 14, but rather it is the altered version of that shared input, namely, \( \theta x_{1,1j} \) that acts as a peer for DMU 14.

Observing Table 5 we note that 13 of the DMUs (1, 5, 6, 10, 11, 13, 16, 21, 23, 25, 32, 39, and 42) are technically efficient, and all technically efficient departments except for department 6 are said to be strongly efficient, meaning that they lie on the frontier proper, and exhibit zero slacks. That is, departments 1, 5, 10, 11, 13, 16, 21, 23, 25, 32, 39, and 42 exhibit strong technical efficiency, while department 6 has positive slacks on certain dimensions, meaning that it would be declared as weakly efficient.

6. Conclusions

This paper has examined the situation in DEA when DMUs are not necessarily independent of one another. Specifically, we consider the case of groups of departments in a university sharing a common input. Unlike the case where an input that is shared by a group of units and can be split into segments and allocated to those units, our example is one where the resource cannot reasonably be split and must be viewed as shared in its entirety among the group members. A similar phenomenon occurs when a set of DMUs, such as maintenance crews, fall under the jurisdiction of a district, and where all crews in a district share non-divisible resources held by the district. Another example would be the grouping of automobile dealerships that benefit from block ads.

An interesting research problem not covered by the model presented herein is that where DMUs can be grouped according to a set of criteria rather than simply a single criterion. For example, rather than automobile dealerships being grouped only according to a specific set of block ads, suppose as well that the dealerships could be further grouped by different factors such as the number of direct competitors in the area where dealerships are located. This is a topic for future research.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

Sonia Avilés-Sacoto acknowledges the financial support from Poligrant No.10099 from USFQ. This research is also supported by the National Natural Science Funds of China (No. 71828101) and the Natural Sciences and Engineering Research Council of Canada (No. A8966).

References


Tsai, P. F., & Molinero, C. (2002). A variable returns to scale data envelopment analysis model for the joint
