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A modified super-efficiency DEA model for infeasibility

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The super-efficiency data envelopment analysis (DEA) model is obtained when a decision making unit (DMU) under evaluation is excluded from the reference set. This model provides for a measure of stability of the "efficient" status for frontier DMUs. Under the assumption of variable returns to scale (VRS), the super efficiency model can be infeasible for some efficient DMUs, specifically those at the extremities of the frontier. The current study develops an approach to overcome infeasibility issues. It is shown that when the model is feasible, our approach yields super-efficiency scores that are equivalent to those arising from the original model. For efficient DMUs that are infeasible under the super-efficiency model, our approach yields optimal solutions and scores that characterize the extent of super-efficiency in both inputs and outputs. The newly developed approach is illustrated with two real world data sets.

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1. Introduction

The data envelopment analysis (DEA) model of Charnes *et al* (1978) provides a methodology for evaluating a relative efficiency score for each member of a set of peer decision-making units (DMUs); this constant returns (CRS) model was extended by Banker *et al* (1984) to included variable returns to scale (VRS). An important problem in the DEA literature is that of ranking those DMUs deemed efficient by the DEA model, all of which have a score of unity. One approach to the ranking problem is that provided by the super efficiency model of Andersen and Petersen (1993). See also Banker *et al* (1989). The super efficiency model involves executing the standard DEA models (CRS or VRS) , but under the assumption that the DMU being evaluated is excluded from the reference set. Specifically, the super efficiency score in say the input-oriented model, provides a measure of the proportional increase in the inputs for a DMU that could take place without destroying the 'efficient' status of that DMU relative to the frontier created by the *remaining* DMUs. The super efficiency score can also be thought of as a measure of stability. That is, if input data for instance, is subject to error or change over time, the super efficiency score provides a means of evaluating the extent to which such changes could occur without violating that DMU's status as an efficient unit. Hence, the score yields a measure of stability.

In addition to being a tool for ranking, the super-efficiency concept has been used in other situations, for example, two-person ratio efficiency games (Rousseau and Semple, 1995), and acceptance decision rules (Seiford and Zhu, 1998a), among others.

It is well known that under certain conditions, the super-efficiency DEA model may not have feasible solutions for efficient DMUs (see, eg, Zhu, 1996; Dulá and Hickman, 1997; Seiford and Zhu, 1998a,b, 1999). Given the wide use of the super-efficiency concept, it is thus worthwhile to develop approaches that can overcome this infeasibility problem. This is the scope of the current paper. As shown in Seiford and Zhu (1999), infeasibility must occur in the case of the VRS super-efficiency model. Although infeasibility implies a form of stability in DEA sensitivity analysis (Seiford and Zhu, 1998b), limited efforts have been made to provide numerical super-efficiency scores for those efficient DMUs for which feasible solutions are unavailable in the VRS super-efficiency model. Lovell and Rouse (2003) developed a standard DEA approach to the super-efficiency model by scaling up the inputs (scaling down the outputs) of a DMU under evaluation. As a result, a feasible solution can be found for efficient DMUs that do not have such (feasible) solutions in the standard VRS super-efficiency model. The super-efficiency scores for all efficient DMUs without feasible solutions are then equal to the user-defined scaling factor. Chen (2004, 2005) suggests using both the input- and output-oriented VRS super-efficiency models to quantify the super-efficiency when infeasibility occurs. However, Chen's approach will fail if both the input- and output-oriented VRS super-efficiency models are infeasible.

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The current paper proposes an alternative approach to solve the infeasibility problem in the VRS super-efficiency model. Our approach provides VRS super-efficiency scores that are equivalent to those arising from the VRS super-efficiency model when feasibility is present. When the VRS super-efficiency model is infeasible, our approach determines a (virtual) referent DMU formed by the remaining DMUs and yields a score that characterizes the super-efficiency in inputs and outputs. Since the super-efficiency and standard DEA models always have feasible solutions and yield equivalent results for inefficient DMUs, the current study assumes that a DMU under evaluation is efficient (ie on the DEA frontier).

The rest of the paper is organized as follows. Section 2 presents the existing VRS super-efficiency models. Section 3 develops our new approach. The relationship between our approach and that of Lovell and Rouse (2003) is discussed in Section 4. Section 5 applies the newly developed approach to data on the 20 largest Japanese companies and 15 US cities that are used in Chen (2004). Conclusions are presented in Section 6.

2. Super-efficiency DEA

Suppose we have a set of n DMUs $\{DMU_j : j = 1, 2, \ldots, n\}$. Each DMU_j has a set of s outputs, y_{rj} $(r = 1, 2, \ldots, s)$, and a set of m inputs, x_{ij} $(i = 1, 2, \ldots, m)$. Based upon the VRS DEA model (Banker *et al*, 1984), the input-oriented VRS super-efficiency model can be expressed as

$$
\begin{aligned}
& \min \theta \\
\text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leqslant \theta x_{ik}, \qquad i = 1, 2, \ldots, m \\
& \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geqslant y_{rk}, \qquad r = 1, 2, \ldots, s \\
& \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1 \\
& \theta \geqslant 0 \\
& \lambda_j \geqslant 0, \qquad j \neq k
\end{aligned} \tag{1}
$$

where the DMU_k under evaluation is excluded from the reference set.

Note that model (1) is an input-oriented super-efficiency DEA model. When DMU_k is efficient and model (1) is feasible, $\theta^* > 1$, indicating that DMU_k's inputs are increased to reach the frontier formed by the rest of the DMUs. That is, super-efficiency is expressed in terms of input increases. We can use θ^* as the super-efficiency score for DMU_k.

The output-oriented VRS super-efficiency model can be expressed as

$$
\begin{aligned}
& \max \phi \\
\text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leqslant x_{ik}, \qquad i = 1, 2, \ldots, m \\
& \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geqslant \phi y_{rk}, \qquad r = 1, 2, \ldots, s \\
& \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1 \\
& \phi \geqslant 0 \\
& \lambda_j \geqslant 0, \qquad (j \neq k)
\end{aligned} \tag{2}
$$

When DMU_k is efficient and model (2) is feasible, $\phi^* \leqslant 1$, indicating that DMU_k's outputs are decreased to reach the frontier formed by the rest of the DMUs, that is super-efficiency is represented by output reductions, and the degree of output-oriented super-efficiency is given by $1/\phi^* \geqslant 1$.

These models can be infeasible in certain situations. Seiford and Zhu (1999) provide the necessary and sufficient conditions for the infeasibility of these models. It is shown that in the case that a VRS efficient DMU has the largest output(s) (regardless of the input values), model (1) must be infeasible; if it has the smallest input(s) (regardless of the output values), model (2) must be infeasible. Note that any data set always contains such efficient DMUs. Thus, models (1) and (2) must be infeasible for these efficient DMUs.

Note that when model (1) (or model (2)) is infeasible, no numerical value can be assigned to θ^* (or $1/\phi^*$). The next section will develop an approach to provide a numerical value to the input-oriented (or output-oriented) super-efficiency score.

3. New model

Infeasibility of model (1) (or (2)) occurs when a VRS efficient DMU under evaluation cannot reach the frontier formed by the rest of DMUs via increasing the inputs (or decreasing the outputs). Unlike the standard super efficiency models (1) and (2), each of which has a specific orientation (input or output), our model proposes moving to the frontier by way of projection in both directions. In practical terms, rather than asking, for a given efficient DMU, either how much increase in inputs is possible, *or* how much reduction in outputs is possible, while still retaining its efficient status, our model describes the minimum movement in both directions needed to reach the frontier generated by the remaining DMUs. Viewed another way, in the case of infeasibility, our model derives the minimum change needed to project a data point, classified as an *extremity*, to a non extreme position.

Consider the following model for DMU_k

$$\begin{aligned} \text{Min} \quad & \tau + M \times \beta \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leqslant (1+\tau) x_{ik}, \qquad i = 1, 2, \ldots, m \\ & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geqslant (1-\beta) y_{rk}, \qquad r = 1, 2, \ldots, s \\ & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1 \\ & \beta \geqslant 0, \lambda_j \geqslant 0, \qquad j = 1, 2, \ldots, n;\ j \neq k \end{aligned} \tag{3}$$

where M is a user-defined large positive number. (In our application, M is set equal to 10^5.)

Theorem 1 *Model* (1) *is infeasible* if and only if $\beta^* > 0$, *where* β^* *is the optimal solution in model* (3).

Proof Note that $\beta > 0$. Suppose model (1) is infeasible. If $\beta^* = 0$, this means that model (1) is feasible, which is a contradiction. Therefore, $\beta^* > 0$. Now, suppose $\beta^* > 0$, and suppose model (1) is feasible. This means that $\beta^* = 0$ is a feasible solution to model (3), in contradiction to the fact that $\beta^* > 0$ is optimal. Therefore, model (1) is infeasible. This completes the proof. □

Theorem 1 indicates that model (1) is feasible if and only if $\beta^* = 0$. This further indicates that $1 + \tau^* = \theta^*$, where $(*)$ denotes the optimal values in models (1) and (3). In other words, when model (1) is feasible, model (3) is equivalent to model (1), in the sense that the objective function values of the two models are identical.

Theorem 2 $1 > \beta^* \geqslant 0$ *and* $\tau^* > -1$ *in model* (3).

Proof Note that in model (3), $\lambda_j \geqslant 0$, $j = 1, 2, \ldots, n$, and $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1$. Therefore, $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} > 0$, $r = 1, 2, \ldots, s$. Thus, there must exist $\beta \in [0, 1)$ such that $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geqslant (1-\beta) y_{rk}$, $r = 1, 2, \ldots, s$. Note also that model (3) minimizes β. Therefore, $1 > \beta^* \geqslant 0$. Similarly, we can prove that $\tau^* > -1$. □

From Theorem 2 it follows that when model (1) is infeasible, $1/(1-\beta^*) > 1$ and $1 + \tau^* > 0$, that is, in order to have a feasible solution, DMU_k must decrease its outputs. Further, we define the super-efficiency score as $1 + \tau^* + 1/(1-\beta^*)$ (> 1). This super-efficiency score consists of a component for input super-efficiency, namely $1 + \tau^* > 0$, and a component for output super-efficiency, namely $1/(1-\beta^*) > 1$.

There are two cases associated with this new super-efficiency score for DMUs not having feasible solutions in model (1).

Case 1: $\tau^* > 0$ (or $1 + \tau^* > 1$). In this case DMU_k must increase its inputs and decrease its outputs to reach the frontier formed by the rest of the DMUs. This means that DMU_k exhibits super-efficiency in both inputs and outputs.

Case 2: $\tau^* < 0$ (or $1 + \tau^* < 1$). This indicates that DMU_k must decrease its inputs and outputs to reach the frontier formed by the rest of the DMUs, meaning that DMU_k's super-efficiency is reflected by the outputs only.

In summary, our new model (3) yields a score equivalent to the original VRS super-efficiency score if model (1) is feasible. (When $\beta = 0$, the actual value of the super-efficiency score based upon model (3) is $1 + \tau^* + 1 = \theta^* + 1$, that is, the input super-efficiency component of $1 + \tau^*$ is the original VRS super-efficiency score θ^*.) When model (1) is infeasible, our model (3) determines an optimal solution and yields a numerical super-efficiency score that captures the super-efficiency in both inputs and outputs.

The above discussion is based upon an input-orientation and model (1). For the output-orientation, we have the equivalent model

$$\begin{aligned} \text{Min} \quad & \gamma + M \times \delta \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leqslant (1+\delta) x_{ik}, \qquad i = 1, 2, \ldots, m \\ & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geqslant (1-\gamma) y_{rk}, \qquad r = 1, 2, \ldots, s \\ & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1 \\ & \delta \geqslant 0, \lambda_j \geqslant 0, \qquad j = 1, 2, \ldots, n;\ j \neq k \end{aligned} \tag{4}$$

where M is a user-defined large positive number.

Similar to Theorems 1 and 2, we have

Theorem 3 *Model* (2) *is infeasible if and only if* $\delta^* > 0$.

Theorem 4 $\gamma^* < 1$ *and* $\delta^* > 0$ *in model* (4).

Theorem 3 indicates that when model (2) is feasible, model (4) is equivalent to model (2) in that $1 - \gamma^* = \phi^*$.

When model (2) is infeasible, Theorem 4 indicates that $1/(1-\gamma^*) > 0$, meaning that in order to have a feasible solution, DMU_k must increase its inputs. Further, we define the super-efficiency score for DMUs not having feasible solutions under model (2) as $1 + \delta^* + 1/(1-\gamma^*) (> 1)$.

When $1 > \gamma^* > 0$, DMU_k must decrease its outputs and increase its inputs to reach the frontier formed by the rest of the DMUs, indicating that DMU_k exhibits super-efficiency in both inputs and outputs. When $\gamma^* < 0$, DMU_k must increase both inputs and outputs to reach the frontier formed by the rest of the DMUs, indicating that DMU_k exhibits super-efficiency in inputs only.

4. Discussion

In this section, we examine the relation between our approach and that of Lovell and Rouse (2003). While the discussion is based upon input-oriented models, similar arguments apply to output-oriented models.

The Lovell and Rouse (2003) model is defined as follows:

$$
\begin{aligned}
\text{Min} \quad & \theta \\
\text{s.t.} \quad & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_{ij} + \lambda_k(\alpha \times x_{ik}) \leqslant \theta(\alpha \times x_{ik}), \quad i = 1, 2, \ldots, m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_{rj} + \lambda_k y_{rk} \geqslant y_{rk}, \quad r = 1, 2, \ldots, s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j + \lambda_k = 1 \\
& \lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n
\end{aligned} \tag{5}
$$

where $\alpha > 1$ is a *user-defined* scaling factor. Lovell and Rouse (2003) suggest using $\alpha = \text{Max}_i\ (\text{Max}\ x_{ij}/\text{Min}\ x_{ij}) + 1$. The super-efficiency is then defined as $\theta \times \alpha$.

Model (5) is equivalent to the following model

$$
\begin{aligned}
\text{Min} \quad & \theta \\
\text{s.t.} \quad & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_{ij} \leqslant (\theta - \lambda_k)\alpha x_{ik}, \quad i = 1, 2, \ldots, m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_{rj} \geqslant (1 - \lambda_k) y_{rk}, \quad r = 1, 2, \ldots, s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 - \lambda_k \\
& \lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n
\end{aligned} \tag{6}
$$

Now, let $(\theta - \lambda_k)\alpha = 1 + \tau$, where τ is a variable that is free in sign. Then, we have $\theta = (1 + \tau + \alpha\lambda_k)/\alpha$ and the super-efficiency score can be expressed as $\alpha\theta = 1 + \tau + \alpha\lambda_k$. Because α is a user-defined constant, model (6) is equivalent to the following model

$$
\begin{aligned}
\text{Min} \quad & \tau + \alpha\lambda_k \\
\text{s.t.} \quad & \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j x_{ij} \leqslant (1 + \tau) x_{ik}, \quad i = 1, 2, \ldots, m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j y_{rj} \geqslant (1 - \lambda_k) y_{rk}, \quad r = 1, 2, \ldots, s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 - \lambda_k \\
& \lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n
\end{aligned} \tag{7}
$$

Theorem 5 *In model* (7), *there does not exist any optimal solution such that* $1 > \lambda_k^* > 0$.

Proof Suppose there exists an optimal solution in model (7) such that $1 > \lambda_k^* > 0$. This indicates that the following model has a feasible solution

$$
\begin{aligned}
\text{Min} \quad & \tau \\
\text{s.t.} \quad & \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\lambda_j}{1 - \lambda_k^*} x_{ij} \leqslant \frac{1 + \tau}{1 - \lambda_k^*} x_{ik}, \quad i = 1, 2, \ldots, m \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\lambda_j}{1 - \lambda_k^*} y_{rj} \geqslant y_{rk}, \quad r = 1, 2, \ldots, s \\
& \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\lambda_j}{1 - \lambda_k^*} = 1 \\
& \lambda_j \geqslant 0, \quad j = 1, 2, \ldots, n.
\end{aligned}
$$

This further indicates that model (1) is always feasible, in contradiction. This proves the theorem. □

Theorem 5 indicates that either $\lambda_k^* = 0$ or $\lambda_k^* = 1$. If $\lambda_k^* = 0$, this means that model (1) is feasible. As a result, model (6) is equivalent to model (3), that is, both Lovell and Rouse (2003) and our approaches yield the identical results to the original super-efficiency model (1). If $\lambda_k^* = 1$, Theorem 5 also indicates that model (1) is infeasible. In this case, model (7) yields the optimal solutions of $\lambda_j^* = 0$, $j \neq k$ and $\tau^* = -1$. As a result, Lovell and Rouse's (2003) super-efficiency is equal to the user-defined α for all the efficient DMUs not having feasible solutions in model (1).

We finally note that the value of α has nothing to do with the constraints of model (7) when model (1) is infeasible for DMU_k. In fact, if we change $\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1 - \lambda_k$ to $\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1$, and set α equal to a large enough value, model (6) becomes model (3).

5. Application

We apply models (3) and (4) to two data sets used in Chen (2004). One consists of the 20 largest Japanese companies in 1999 (see Table 1). The other consists of 15 of Fortune's top US cities in 1996 (see Table 2).

5.1. *Japanese companies*

The DEA inputs are assets (million \$), equity (million \$) and number of employees and the DEA output is revenue (million \$). Either model (1) or model (2) indicates that five of them are VRS-efficient (see last two columns in Table 1). DMU1 is infeasible under model (1) and DMU18 is infeasible under model (2).

Table 3 reports the results from models (3) and (4) for the five efficient DMUs. Model (3) yields a super-efficiency score of 2.0161 for DMU1 with $\tau^* = 0.0104$ and $\beta^* = 0.0057$, indicating that DMU1 has super-efficiency in both inputs and outputs. Model (4) yields a super-efficiency score of 3.22798 for DMU18 with $\delta^* = 1.8999$ and $\gamma^* = -2.048$, indicating that DMU18 has super-efficiency in inputs only.

Table 1 Japanese companies

DMU	*Company*	*Asset*	*Equity*	*Employee*	*Revenue*	*Model* (1) θ_o^*	*Model* (2) $1/\phi_o^*$
1	MITSUI & CO.	50905.3	5137.9	40000	106793.2	**infeasible**	**1.01523**
2	ITOCHU CORP.	51432.5	2333.8	5775	106184.1	**6.69295**	**1.3177**
3	MITSUBISHI CORP.	67553.2	7253.2	36000	104656.3	0.74248	0.98068
4	TOYOTA MOTOR CORP.	112698.1	47177	183879	97387.6	0.4108	0.91191
5	MARUBENI CORP.	49742.9	2704.3	5844	91361.7	0.91739	0.89127
6	SUMITOMO CORP.	41168.4	4351.5	30700	86921	**1.02091**	**1.0227**
7	NIPPON TELEGRAPH & TEL.	133008.8	47467.1	138150	74323.4	0.26865	0.69594
8	NISSHO IWAI CORP.	35581.9	1274.4	19461	66144	**1.14580**	**1.14784**
9	HITACHI LTD.	73917	21914.2	328351	60937.9	0.40528	0.57061
10	MATSUSHITA ELECTRIC INDL.	60639	26988.4	282153	58361.6	0.47569	0.54648
11	SONY CORP.	48117.4	13930.7	177000	51903	0.54156	0.51337
12	NISSAN MOTOR	52842.1	9583.6	39467	50263.5	0.47975	0.4707
13	HONDA MOTOR	38455.8	13473.8	112200	47597.9	0.62931	0.59028
14	TOSHIBA CORP.	46013	8023.3	198000	40492.7	0.45933	0.41827
15	FUJITSU LTD.	39052.2	8901.6	188000	40050.3	0.53631	0.48833
16	TOKYO ELECTRIC POWER	110055.8	12157.7	50558	38869.5	0.18567	0.36397
17	NEC CORP.	38015	6517.4	157773	36356.4	0.50901	0.45666
18	TOMEN CORP.	16696	676.1	3654	30205.3	**2.89988**	**infeasible**
19	JAPAN TOBACCO	17023.6	10816.6	31000	29612.2	0.98076	0.9563
20	MITSUBISHI ELECTRIC CORP.	31997	4129.6	116479	28982.2	0.5218	0.44136

The bold numerals enable easy identification of items discussed in the paper.

Table 2 US cities

DMU	*City*	*Houseprice*	*Rental*	*Violent*	*Income*	*B. Degree*	*Doctor*	*Model* (1) θ_o^*	*Model* (2) $1/\phi_o^*$
1	Seattle	586	581	1193.06	46928	0.6534	9.878	**1.44335**	**1.0934**
2	Denver	475	558	1131.64	42879	0.5529	5.301	**1.01593**	**1.0527**
3	Philadelphia	201	600	3468	43576	1.135	18.2	**infeasible**	**infeasible**
4	Minneapolis	299	609	1340.55	45673	0.729	7.209	**1.22752**	**1.086**
5	Raleigh	318	613	634.7	40990	0.319	4.94	**1.16766**	**infeasible**
6	StLouis	265	558	657.5	39079	0.515	8.5	**1.51628**	**infeasible**
7	Cincinnati	467	580	882.4	38455	0.3184	4.48	0.94968	0.897
8	Washington	583	625	3286.7	54291	1.7158	15.41	**infeasible**	**1.5344**
9	Pittsburgh	347	535	917.04	34534	0.4512	8.784	**1.04529**	**infeasible**
10	Dallas	296	650	3714.3	41984	1.2195	8.82	0.92652	0.9532
11	Atlanta	600	740	2963.1	43249	0.9205	7.805	0.77243	0.8137
12	Baltimore	575	775	3240.75	43291	0.5825	10.05	0.73827	0.8009
13	Boston	351	888	2197.12	46444	1.04	18.208	**infeasible**	**1.3181**
14	Milwaukee	283	727	778.35	41841	0.321	4.665	**1.06559**	**1.0276**
15	Nashville	431	695	1245.75	40221	0.2365	3.575	0.80117	0.873

The bold numerals enable easy identification of items discussed in the paper.

Table 3 Results for Japanese companies

		Input-oriented				*Output-oriented*			
DMU	*Company*	*Super-efficiency*	τ^*	β^*		*Super-efficiency*	δ^*	γ^*	
1	MITSUI &CO.	**2.0161***	0.0104	0.0057	$\lambda_2 = 1$	1.01523	0	0.015	$\lambda_2 = 0.9486$, $\lambda_6 = 0.0514$
2	ITOCHU CORP.	6.693	5.693	0	$\lambda_1 = 0.9605$, $\lambda_5 = 0.0395$	1.3177	0	0.2411	$\lambda_5 = 0.8109$, $\lambda_8 = 0.0218$, $\lambda_{18} = 0.1673$
6	SUMITOMO CORP.	1.0209	0.0209	0	$\lambda_1 = 0.7405$, $\lambda_{18} = 0.2595$	1.0227	0	0.0222	$\lambda_1 = 0.7154$, $\lambda_{18} = 0.2846$
8	NISSHO IWAI CORP.	1.1458	0.1458	0	$\lambda_2 = 0.473$, $\lambda_{18} 0.527$	1.14784	0	0.1288	$\lambda_2 = 0.3609$, $\lambda_{18} = 0.6391$
18	TOMEN CORP.	2.8999	1.8999	0	$\lambda_2 = 0.6477$, $\lambda_8 = 0.3523$	**3.22798***	1.8999	−2.048	$\lambda_2 = 0.6477$, $\lambda_8 = 0.3523$

*This indicates that the original super-efficiency model is infeasible.

Table 4 Results for US cities

		Input-oriented				Output-oriented			
DMU	*City*	*Super-efficiency*	α^*	β^*		*Super-efficiency*	δ^*	γ^*	
3	Philadelphia	**2.8930***	0.8759	0.0168	$\lambda_8=0.1123, \lambda_{13}=0.8877$	**3.5225***	0.3184	0.5463	$\lambda_6=1$
5	Raleigh	1.1677	0.1677	0	$\lambda_6=0.3081, \lambda_{14}=0.6919$	**2.08478***	0.0359	0.0466	$\lambda_6=1$
6	StLouis	1.5163	0.5163	0	$\lambda_1=0.2613, \lambda_4=0.0921, \lambda_5=0.1105, \lambda_9=0.5362$	**2.7669***	0.1519	0.3808	$\lambda_3=0.0299, \lambda_5=0.7057, \lambda_{14}=0.2644$
9	Pittsburgh	1.0453	0.0453	0	$\lambda_3=0.0293, \lambda_6=0.9707$	**2.0764***	0.0430	0.0323	$\lambda_6=1$
13	Boston	**2.5908***	0.5653	0.0249	$\lambda_3=0.8403, \lambda_8=0.1597$	1.31804	0	0.2413	$\lambda_3=0.5478, \lambda_6=0.4522$

*This indicates that the original super-efficiency model is infeasible.

Table 3 also reports the benchmarks as indicated by the non-zero λ^*. It can be seen that DMU1 is benchmarked against DMU2 by model (3) and DMU18 is benchmarked against the convex combination of DMUs 2 and 8 by model (4).

Finally, note that $\beta^*=\delta^*=0$ when model (1) (model (2)) is feasible.

5.2. US cities

The data set for the 15 US cities has three inputs, namely, high-end housing price (1,000 US$), lower-end housing monthly rental (US$), and number of violent crimes, and three outputs, namely, median household income (US$), number of bachelor's degrees (million) held by persons in the population, and number of doctors (thousand).

The last two columns report the super-efficiency scores from models (1) and (2). It can be seen that 10 cities are efficient. There are seven infeasibility cases. Table 4 reports the results from models (3) and (4) for these seven cases. Note that DMU3 (Philadelphia) is infeasible under both models (1) and (2). Our model (3) yields a super-efficiency score of 2.893 (with $\tau^*=0.8759$ and $\beta^*=0.0168$) and model (4) yields a super-efficiency score of 3.5225 (with $\delta^*=0.3184$ and γ^*=0.5463). While model (3) benchmarks DMU3 against DMUs 8 and 13, model (4) benchmarks DMUs against DMU6 (see columns 6 and 10).

The results in Table 4 indicate that all cities having no feasible solutions in model (1) or (2) have super-efficiency in both inputs and outputs, as indicated by $\tau^*>0$ and $\gamma^*>0$.

6. Conclusions

The current paper develops a modified super-efficiency DEA model to overcome the infeasibility issue under the assumption of VRS. The newly developed approach yields (i) an optimal solution and a super-efficiency score for efficient DMUs for which feasible solutions do not exist under the original super-efficiency model; and (ii) super-efficiency scores that are equivalent to those from the original super-efficiency model when feasible solutions do exist. To some extent, the DEA Malmquist productivity index (Färe and Grosskopf, 1992), and the DEA benchmarking models (Cook *et al*, 2004) have similar structures to that of the super-efficiency model. The current study, therefore, offers the possibility to extend the Malmquist productivity index and DEA benchmarking models into situations where the non-CRS assumption of the DEA frontier is required.

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