Setting scale efficient targets in DEA via returns to scale estimation method

Introduction

Appa and Yue\(^1\) proposed a method for setting scale efficient targets in data envelopment analysis (DEA). In the presence of possible multiple optimal solutions, their best returns to scale (RTS) model yields the same scale efficient targets under both input-oriented and output-oriented DEA models. They indicate that their unique scale efficient targets correspond to the largest most productive scale size (MPSS). Based upon RTS estimation method, this paper develops an alternative approach for setting the scale efficient targets which can correspond to either the largest MPSS or the smallest MPSS.

Data envelopment analysis (DEA) has been proved to be a useful tool for evaluating the relative efficiency of peer decision making units (DMUs) which produce multiple outputs by consuming multiple inputs. Although DEA is originally developed to measure the technical or mix (technical and scale) efficiency, it has been modified to characterise returns to scale (RTS) classification/scale efficiency.\(^2\) DEA determines a unique best-practice frontier. However, because of the possible multiple optimal solutions/orientation of the DEA models, scale efficient targets on the best-practice frontier for inefficient DMUs may not be uniquely determined. Appa and Yue\(^1\) developed a DEA-based method for setting unique scale efficient targets in DEA. In the presence of possible multiple optimal solutions, their approach yields the same scale efficient target for DMU under both input-oriented and output-oriented DEA models. They show that their scale targets are related to the largest most productive scale size (MPSS).\(^3\) However, in order to identify targets corresponding to the smallest MPSS, an additional constraint is needed.

This paper shows that in fact, these unique scale efficient targets can be obtained directly from MPSS concept even if multiple optimal solutions (or multiple MPSS) are present. It is shown that these scale efficient targets are obtained from linear programming problems for determining the RTS classification. Since the scale efficiency is related to the RTS classification, it is desirable that the targets are determined via RTS estimation method.

Setting scale efficient targets

In this section, we show that unique scale efficient targets determined by Appa and Yue’s approach\(^1\) can actually be obtained from RTS estimation method based upon MPSS concept. Let us first review a RTS estimation method based upon an input-oriented CRS (constant RTS) model.\(^4\) Suppose we have \(n\) DMUs. DMU\(_j\) = 1, 2, \ldots, \(n\) produces \(s\) different outputs, \(y_o(r = 1, 2, \ldots, s)\), using \(m\) different inputs, \(x_i(i = 1, 2, \ldots, m)\). Then the CRS model can be written as

\[
\min \theta - \epsilon \left( \sum_{j=1}^{m} x_j^r + \sum_{r=1}^{s} y^r \right)
\]

subject to

\[
\begin{align*}
\sum_{j=1}^{n} \lambda_j x_{io} + s_{io}^r &= \theta x_{io} & i &= 1, 2, \ldots, m; \\
\sum_{j=1}^{n} \lambda_j y_{or} - s_{or}^r &= y_{ro} & r &= 1, 2, \ldots, s; \\
\lambda_j, s_{io}^r, s_{or}^r &\geq 0.
\end{align*}
\]

Where, \(x_{io}\) and \(y_{ro}\) are respectively the \(i\)th and \(r\)th output for DMU\(_o\) under evaluation. Association with the \(m + s\) input and output constraints, some non-zero slacks (\(s_{io}^r, s_{or}^r\)) may be identified by utilizing a two-stage process where the efficiency score (\(\theta^*\)) is first calculated and then the sum of slacks is maximised.

On the basis of optimal solutions \(\lambda_j^*\) to (1), we have the following RTS method which is relative to the concept of MPSS.

(i) If \(\sum_{j=1}^{n} \lambda_j^* = 1\) in any alternate optima then CRS prevail on DMU\(_o\).

(ii) If \(\sum_{j=1}^{n} \lambda_j^* < 1\) for all alternate optima then IRS (increasing RTS) prevail on DMU\(_o\).

(iii) If \(\sum_{j=1}^{n} \lambda_j^* > 1\) for all alternate optima then DRS (decreasing RTS) prevail on DMU\(_o\).

It has been noted that in practice, above RTS estimation method fail when model (1) has alternate optima.\(^5,6\) Since it may be impossible or at least unreasonable to generate all possible multiple optima in many real world applications, a number of modifications or extensions or the above RTS estimation method have been developed to deal with multiple optima.\(^2\) Banker et al\(^7\) provided the following linear programming model (given \(\sum_{j=1}^{n} \lambda_j^* < 1\) obtained from (1))

\[
\max \sum_{j=1}^{n} \lambda_j^* - \epsilon \left( \sum_{j=1}^{m} \lambda_j + \sum_{r=1}^{s} y^r \right)
\]
(ii) Given the existence of an optimal solution with \( P \), the possible multiple optimal solutions in (1).  

(i) Given the existence of an optimal solution with \( P \), we obtain an optimal solution with \( \sum_{j=1}^{n} \hat{\lambda}_j > 1 \) in (1), then we change the objective of (2) to minimisation and replace \( \sum_{j=1}^{n} \hat{\lambda}_j \leq 1 \) with \( \sum_{j=1}^{n} \hat{\lambda}_j > 1 \) and denote the corresponding optimal value of \( \sum_{j=1}^{n} \hat{\lambda}_j \) by \( \sum_{j=1}^{n} \hat{\lambda}_j^* \).

On the basis of model (2), we then have the following modified RTS estimation method which is not affected by the possible multiple optimal solutions in (1).

(i) Given the existence of an optimal solution with \( \sum_{j=1}^{n} \hat{\lambda}_j < 1 \) in (1), the RTS at DMU \( o \) are CRS if and only if \( \sum_{j=1}^{n} \hat{\lambda}_j = 1 \), and IRS if and only if \( \sum_{j=1}^{n} \hat{\lambda}_j < 1 \).

(ii) Given the existence of an optimal solution with \( \sum_{j=1}^{n} \hat{\lambda}_j > 1 \) in (1), the RTS at DMU \( o \) are CRS if and only if \( \sum_{j=1}^{n} \hat{\lambda}_j = 1 \), and DRS if and only if \( \sum_{j=1}^{n} \hat{\lambda}_j > 1 \).

As a matter of fact, model (2) and MPSS concept can be used to determine the unique scale efficient target for DMU \( o \) relative to the largest MPSS or the smallest MPSS. To obtain the largest MPSS, we re-write model (2) as the following linear programming model.

\[
\begin{align*}
\min \sum_{j=1}^{n} \hat{\lambda}_j - \delta \left( \sum_{i=1}^{n} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\
\text{subject to} \quad \sum_{j=1}^{n} \hat{\lambda}_j x_{ij} + s_i^- = \theta^* x_{io} & \quad i = 1, 2, \ldots, m; \\
\sum_{j=1}^{n} \hat{\lambda}_j y_{rj} - s_r^+ = y_{ro} & \quad r = 1, 2, \ldots, s; \\
\hat{\lambda}_j, s_i^-, s_r^+ \geq 0.
\end{align*}
\]

(3)

On the basis of optimal values from (3), that is, \( s_i^-, s_r^+ \) and \( \Sigma \hat{\lambda}_j \), the MPSS concept yields the following scale-efficient target for DMU \( o \) corresponding to the largest MPSS.

\[
\begin{align*}
\text{MPSS}_{\text{max}} : \quad \hat{x}_{io} &= (\theta^* x_{io} - \Sigma s_i^-) / \Sigma \hat{\lambda}_j \\
\hat{y}_{ro} &= (y_{ro} + \Sigma s_r^+) / \Sigma \hat{\lambda}_j
\end{align*}
\]

(4)

where \( (\sim) \) represents the target value.

If we change the objective of (3) to maximisation,

\[
\begin{align*}
\max \sum_{j=1}^{n} \hat{\lambda}_j + \delta \left( \sum_{i=1}^{n} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\
\text{subject to} \quad \sum_{j=1}^{n} \hat{\lambda}_j x_{ij} + s_i^- = \theta^* x_{io} & \quad i = 1, 2, \ldots, m; \\
\sum_{j=1}^{n} \hat{\lambda}_j y_{rj} - s_r^+ = y_{ro} & \quad r = 1, 2, \ldots, s; \\
\hat{\lambda}_j, s_i^-, s_r^+ \geq 0.
\end{align*}
\]

(5)

then we have the scale efficient target corresponding to the smallest MPSS.

\[
\begin{align*}
\text{MPSS}_{\text{min}} : \quad \hat{x}_{io} &= (\theta^* x_{io} - \Sigma s_i^-) / \Sigma \hat{\lambda}_j \\
\hat{y}_{ro} &= (y_{ro} + \Sigma s_r^+) / \Sigma \hat{\lambda}_j
\end{align*}
\]

(6)

where \( s_i^- \), \( s_r^+ \) and \( \Sigma \hat{\lambda}_j \) are optimal values from model (6) and \( \delta^* \) is the optimal value to (1).

Obviously, \( \text{MPSS}_{\text{max}} \) and \( \text{MPSS}_{\text{min}} \) are uniquely determined by \( \theta^* \) and \( \Sigma \hat{\lambda}_j \), since \( \theta^* \) and \( \Sigma \hat{\lambda}_j \) are optimal values to (1), (3) and (5), as in Appa and Yue.1 We also do not consider the non-uniqueness caused by multiple optimal solutions on slacks. Note that all the models previously developed are input-oriented. However, by using the relationship between an input-oriented CRS model and an output-oriented CRS model,8 it is trivial to show that \( \text{MPSS}_{\text{max}} \) (\( \text{MPSS}_{\text{min}} \)) remains the same under both orientations.

Appa and Yue1 use \( \tau \) (an indicator for scale change) and \( \gamma \) (an indicator for input proportional change after the scale change of \( \tau \), see Appa and Yue1 for a complete discussion on \( \tau \) and \( \gamma \)) to set the scale efficient target via a best RTS model. As a matter of fact, if the target determined by the best RTS model corresponds to the largest MPSS (\( \text{MPSS}_{\text{max}} \)), then we have \( \tau = 1 / \Sigma \hat{\lambda}_j \) and \( \gamma = 1 / \Sigma \hat{\lambda}_j (1 - \theta^*) \), where \( \Sigma \hat{\lambda}_j \) is the optimal value in (3) and \( \theta^* \) is the optimal value in (1).

While our approach can obtain a scale efficient target relative to the smallest MPSS via model (5), Appa and Yue1 have to introduce the following additional constraint into their best RTS model.

\[
\gamma = \tau (1 - \theta^*)
\]

(7)

If we use the optimal values in (5) and (7) this can be expressed as \( \gamma = 1 / \Sigma \hat{\lambda}_j (1 - \theta^*) \).

Compared to Appa and Yue’s approach,1 our approach can be viewed as a by-product of RTS estimation. As a result, one can select the largest or the smallest MPSS target for a particular DMU under consideration on the basis of the RTS preference over performance improvement. For example, one may select the smallest MPSS for an IRS DMU and the largest MPSS for a DRS DMU.

Finally, we apply our approach to the 9 DMUs in Appa and Yue.1 Table 1 shows the data and Table 2 reports the
results. It can be seen that our MPSS_{max} targets correspond to the targets identified by the best RTS model, small difference exists in the targets for DMU9, because \theta^* should be equal to 20/23 rather than 0.8796 as in Appa and Yue. Only one exception is found for DMU3. The scale efficient target for DMU3 generated by the best RTS model corresponds to the smallest MPSS. This is because DMU3 is not included in the set of J_{CRS} in Appa and Yue. Otherwise, if DMU3 is included in J_{CRS} then the best RTS model will yield a scale efficient target corresponding to the largest MPSS. Furthermore, note that DMU3 is a weakly efficient DMU with non-zero slack and belongs to the set F of Charnes et al. The set J_{CRS} in Appa and Yue is composed from DMUs in set E and set E’ of Charnes et al. Therefore, in order to establish the one-to-one correspondence between the best RTS model and the largest MPSS, set J_{CRS} should include all DMUs in sets E, E’ and F, that is DMUs with \theta^* = 1 in (1), Appa and Yue (p 64) claims that one does not need to solve model (1) for all DMUs, because if \lambda^*_j is positive then \lambda_{j} \in J_{CRS}. However, sometimes, F DMUs may also be associated with positive optimal lambda values. As a result, one has to solve model (1) for each DMU in order to determine set J_{CRS}. However, if set J_{CRS} includes DMUs in set E, E’ and F as suggested by the current paper, their proposal works.

Conclusions
This current paper develops an alternative approach for setting scale efficient targets in DEA to Appa and Yue’s. Since our approach is based upon RTS estimation method, both RTS classification and scale efficient target can be determined at the same time without additional effort. Like the approach in Appa and Yue, our approach gives the same scale efficient target under both input- and output-oriented DEA models. Also, our approach can yield two scale efficient targets corresponding to the largest and smallest MPSS respectively. This allows the determination of appropriate scale efficient targets for inefficient DMUs when the user has a RTS preference over performance improvement.

References