# Some models and measures for evaluating performances with DEA: past accomplishments and future prospects 

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#### Abstract

This paper covers some of the past accomplishments of DEA (Data Envelopment Analysis) and some of its future prospects. It starts with the "engineering-science" definitions of efficiency and uses the duality theory of linear programming to show how, in DEA, they can be related to the Pareto-Koopmans definitions used in "welfare economics" as well as in the economic theory of production. Some of the models that have now been developed for implementing these concepts are then described and properties of these models and the associated measures of efficiency are examined for weaknesses and strengths along with measures of distance that may be used to determine their optimal values. Relations between the models are also demonstrated en route to delineating paths for future developments. These include extensions to different objectives such as "satisfactory" versus "full" (or "strong'") efficiency. They also include extensions from


[^0]"efficiency" to "effectiveness" evaluations of performances as well as extensions to evaluate social-economic performances of countries and other entities where "inputs" and "outputs'" give way to other categories in which increases and decreases are located in the numerator or denominator of the ratio (=engineering-science) definition of efficiency in a manner analogous to the way output (in the numerator) and input (in the denominator) are usually positioned in the fractional programming form of DEA. Beginnings in each of these extensions are noted and the role of applications in bringing further possibilities to the fore is highlighted.

Keywords Efficiency • Effectiveness • Social Indicators • Engineering-science • Welfare economics • Distance measures

## 1 Introduction

This paper covers some of the past accomplishments and future prospects for DEA (Data Envelopment Analysis). Attempts at covering past accomplishments usually take the form of bibliometric studies. See, for instance, the bibliometric study by Emrouznejad et al. (2007). Entitled "A Bibliography of Data Envelopment Analysis (1978-2003)" it lists more than 3,200 papers, books, etc., written by more than 1,600 authors with a great variety of applications reported in more than 42 countries. This occurred over the 25 year period 1978-2003 starting with publication of the article by Charnes et al. (1978), as indicated by the title of this bibliography, and more progress continues to occur at an increasing pace. See also Gattoufi et al. (2004).

Here we take a different approach and try to bring together some of the ideas as well as some of the models and
methods that have now been developed to deal with problems in DEA formulations that have been, or might be, encountered. Thus, in the next section, Sect. 2, we describe how concepts embodied in two different definitions of efficiency and directed to very different types of problems can be joined together by DEA to increase the power and scope of both. We discuss an example that uses the engi-neering-science definition of efficiency, as in ratios of output (e.g., work) to input (e.g., energy), that are used to evaluate the relative efficiencies of jet aircraft engines. See Mattingly (1996). We show how this one-output-to-oneinput ratio formulation can be extended to multiple outputs and multiple inputs by recourse to the concept of "Pareto efficiency" as used in welfare economics. This is accomplished by using a 'fractional programming'" DEA formulation, which embodies (and generalizes) the engineering-science definition. We then describe how this fractional program gives way to an equivalent linear programming model with a dual that effects the evaluations in a manner that embodies the engineering-science definition in the 'multiplier model'" and the Pareto (or Pareto-Koopmans) definitions of efficiency in the dual or "envelopment model."

In Sect. 3, this discussion is accorded mathematical form in a way that gives rise to the well known "Farrell", and '"CCR'" models of DEA. Weaknesses (as well as strengths) in both models are noted for further development by reference to definitions of "strong'" (or 'full'") and 'weak" efficiency. To assist in these further developments, the next section of this paper (Sect. 4) characterizes the Farrell (1957) measure in terms of its relations to mathematical definitions of distance measures. The Farrell 'radial'" measure is then treated as a ratio of two measures of distance that accommodate an infinite number of such measures, all yielding the same value for their ratio. This is also interpreted in terms of the "units invariance" property of the Farrell measure. However, the Farrell measure is not 'complete" since it does not comprehend all the inefficiencies that the model can identify. The way this weakness is addressed by the CCR model is then noted, but, as is also noted, uses of this CCR model are accompanied by weaknesses in the treatment of non-zero slacks that result in measures that are not units invariant.

Section 5 turns attention to the 'additive model'" as a natural evolute from the CCR model. This model is inclusive but suffers from other difficulties such as a lack of units invariance. In addition it may not be suitable for use in identifying "benchmark'" DMUs to be used in the evaluations of DMUs because this model may identify reference sets that are far removed from the DMU being evaluated. See Coelli (1998).

Section 6 treats other measures and models that have now been developed, which are all shown to be related to the additive model. It starts with the 'Slacks Based Model" (SBM) introduced by Tone (2001) to deal with the lack of units invariance in the additive model measure. Attention is turned to the "Enhanced Russell Measure" (ERM) model of Pastor et al. (1999). After showing the equivalence of SBM and ERM the discussion describes some of the troubles that can be encountered in dealing with non-positive data-such as may be encountered with "profits" and 'losses," say, as outputs. A model that can deal with this problem, because it is "translation invariant" as well as units invariant, is the RAM (Range Adjusted Measure) model of Cooper et al. (2001). In fact, it is shown here that RAM is also 'affine invariant'"-e.g., as in transformations from Centigrade to Fahrenheit units of temperature.

The RAM model also involves the adjunction of a convexity condition. This might lead to a discussion of the BCC model of Banker et al. (1984) and its use in returns-to-scale evaluations. However, this is not discussed here because this topic was recently surveyed in Chapter 2 of Cooper et al. (2004) where reference is also made to the use of "multiplicative" models in which 'exact'" (=numerically valued) elasticities of scale are shown to be determinable without dependence on analyticity properties, like those associated with the use of partial derivatives, to determine their values.

Finally, Sect. 7 is directed to some of the many prospects that are now available for future developments in DEA.

## 2 DEA and extensions

As is well known, the basic ideas in DEA were first developed and applied to empirical data in the pathbreaking article by Farrell (1957). Intended to correct deficiencies found in productivity indexes, Farrell's work actually replaced the concept of productivity with the more general concept of "relative efficiency." Based on the 'activity analysis'" literature of Debreu (1951), and Koopmans (1951), Farrell's uses of these ideas were accorded very little attention until after the publication by Charnes et al. (1978) which extended and increased the scope, the power and the computational convenience of DEA in a manner that we will try to describe. This will include drawing a distinction between the "weakly efficient'" performances, to which the Farrell and Debreu measures were limited, and some of the "fully efficient"" or "strongly efficient" characterizations that are now available.

We start by noting that Charnes et al. (1978) introduced a "fractional programming'" formulation in which the ratios were restricted to lie between the values of zero and unity. This formulation extended the usual engineeringscience single output-to-single input ratio measure of efficiency to multiple inputs and outputs without requiring recourse to a priori prescribed weights, such as are customarily used in engineering to deal with such cases. This fractional (=ratio) formulation provided access to the Charnes and Cooper (1962) transformation that has led to the area of research now known as 'fractional programming." For a description of this area of research and its uses see Schaible (1996). This transformation results in a linear programming problem that is dual to the problem formulated by Farrell. This in turn provides contact with the concept of "Pareto (1909) optimality," as used in "welfare economics," that was later extended for use in "production economics" for "efficiency characterizations"' by Koopmans (1951). It is therefore now referred to as Pareto-Koopmans efficiency-viz.,

Definiton 1 Pareto-Koopmans efficiency is achieved by a DMU (Decision Making Unit) if and only if it is not possible to improve any of its inputs or outputs without worsening some of its other inputs or outputs.

This definition means (1) it is not necessary to assign relative weights, etc., to determine the relative importance of the different inputs and outputs and (2) its fulfillment is a necessary condition for optimality in any system where the weights that might be assigned are all positive.

This fractional programming formulation has also provided a previously unknown contact between the efficiency concepts of welfare economics and of engineering and science. This extends the power and scope of both. Bulla et al. (2000) provide an example involving the relative evaluations of 29 jet aircraft engines that compares DEA with engineering measures. The commonly used engineering measures of efficiency for these engines consist of the ratio of "work rate output" to "fuel energy input." Using this same input and output, DEA produced the same efficiency rankings as the engineering measure. When DEA was expanded to three inputs and two outputs, however, the results were quite different. DEA also offered additional advantages such as information on the sources and amounts of inefficiencies-information that was not available from the engineering measure-as well as "dual evaluators" to estimate the effects of changes in the constraints. See Table 2 in Bulla et al. (2000).

Further implications of this use of DEA can be described as follows. The computations were executed not by the ratio form, but by the linear programming equivalent derived from the Charnes-Cooper transformation of frac-
tional programming. Hence, the evaluations were made by means of the Pareto-Koopmans definition of efficiency, as in welfare economics, but were interpreted in terms of their engineering-science equivalents represented by the corresponding fractional form. Thus both definitions of efficiency were brought into play and, as should be evident, this two-way contact can also make it possible for the engineering-science definition of efficiency to be applied to propositions in welfare economics, if desired, as has now been done in many of the DEA applications that have been reported.

Pareto did not supply anything more than a criterion for choosing between prescribed alternatives. Koopmans extended these possibilities by introducing the concept of "efficiency prices" which provide a basis for determining the 'opportunity costs'" of extending a proposed change to additional alternatives. This kind of extension can be brought to bear in uses such as "compensating variations,", as they are called in economics, that can further advance social welfare by using potential gains to compensate persons who might otherwise be worsened by the proposed changes.

Koopmans restricted his 'efficiency prices" for use only with "final goods" as he termed the "outputs." However, as shown in Chapter 9 of Charnes and Cooper (1961) these efficiency prices can be regarded as 'dual evaluators", available from the duality relations of linear programming. Indeed, as shown in this same Chapter 9, the activity analysis approach is a special case of linear programming so that the full range of possibilities for uses of duality and the computational powers of linear programming become available. For instance, embedding "efficiency prices" in these duality relations makes it possible to extend these "opportunity cost'" evaluations so that inputs as well as outputs can be brought into play without recourse to 'unit costs,'" or like data requirements. See Dantzig (1963) pp. 265-275 for a description of various uses of these duality relations.

Still more possibilities are opened for exploitation along this route. In DEA the dual variable values (=multipliers) take the form of per unit increments in efficiency but other extensions and modifications can be made. In addition, and more importantly, the resulting evaluations are not restricted to the ceteris paribus (all else held constant) characterizations that are usually used in economics. Instead, they are associated with all of the changes needed to optimally adjust all inputs and outputs to any changes that may be proposed for particular subsets of inputs or outputs. To borrow from Robinson (1933), these are mutatis mutandis adjustments so that, step by step, all changes are accompanied by optimal adjustments in all other inputs and outputs. See Cooper et al. (2000) for a use of this concept that extends the usual elasticities and rates
of substitution (ceteris paribus) approaches in economics to more general mutatis mutandis approaches.

Another example of advantages from extensions to inputs as well as outputs may be found in Cooper et al. (2006) where the duality relations of linear programming are described for use in extending the aggregation of efficiency measures to situations in which resource transfers can be made from one DMU to another in order to increase the efficiency of the system. In this way the attainment of full efficiency by all DMUs becomes a necessary but not a sufficient condition for efficiency of the system. In this aggregation of efficiency measures the extension to inputs as well as to outputs is crucial. See, for instance, the "proofs", by Blackorby and Russell (1999), which are directed to showing that satisfactory aggregation measures cannot be achieved in DEA. However, the aggregation in Cooper et al. (2006) succeeds because Blackorby and Russell confined themselves to measures that are incomplete.

## 3 Fractional and linear programming forms

We now give this development precise mathematical form. We start with the fractional programming formulation which (1) generalizes the one-output-to-one-input ratio of engineering and science and (2) derives its efficiency evaluations for each DMU relative to the performances of all DMUs. The model we use is
$\max h_{o}(u, v)=\sum_{r} u_{r} y_{r o} / \sum_{i} v_{i} x_{i o}$
subject to
$\sum_{r} u_{r} y_{r j} / \sum_{i} v_{i} x_{i j} \leq 1$ for $j=1, \ldots, n$,
with
$\frac{u_{r}}{\sum_{i=1}^{m} v_{i} x_{i o}}, \frac{v_{i}}{\sum_{i=1}^{m} v_{i} x_{i o}} \geq \varepsilon>0$,
where $y_{r j}$ and $x_{i j}$ are observed values of outputs and inputs, $r=1, \ldots, s$, and $i=1, \ldots, m$, for each of $j=1, \ldots, n$ DMUs (Decision Making Units) and the $y_{r o}$ and $x_{i o}$ in the objective function represent the outputs and inputs for the DMUo to be evaluated. Here $\varepsilon$ is a non-Archimedean element smaller than any positive real number. See Arnold et al. (1997) for the relation of this non-Archimedean element to 'nonstandard" mathematics. This use of $\varepsilon>0$ guarantees that solutions will be positive in all variables so that "some" worth, however small, will be accorded to each input and output.

Problem (1) is nonlinear and non convex. However, we can apply the Charnes and Cooper (1962) transformation of variables and convert it to an equivalent linear programming problem:
$\max z=\sum_{r=1}^{s} \mu_{r} y_{r o}$
subject to

$$
\begin{align*}
& \sum_{r=1}^{s} \mu_{r} y_{r j}-\sum_{i=1}^{m} v_{i} x_{i j} \leq 0, j=1, \ldots, n  \tag{2}\\
& \sum_{i=1}^{m} v_{i} x_{i o}=1 \\
& \mu_{r}, v_{i} \geq \varepsilon>0
\end{align*}
$$

for which the LP dual is
$z_{o}^{*}=\min \theta-\varepsilon\left(\sum_{i=1}^{m} s_{i}^{-}+\sum_{r=1}^{s} s_{r}^{+}\right)$
subject to
$\sum_{j=1}^{n} x_{i j} \lambda_{j}+s_{i}^{-}=\theta x_{i o} \quad i=1,2, \ldots, m ;$

$$
\begin{align*}
& \sum_{j=1}^{n} y_{r j} \lambda_{j}-s_{r}^{+}=y_{r o} \quad r=1,2, \ldots, s  \tag{3}\\
& \lambda, s_{i}^{-}, s_{r}^{+} \geq 0 \quad \forall i, j, r
\end{align*}
$$

Notice now that both (1) and (2) incorporate the engi-neering-science definition of efficiency with (2) evaluating the denominator in the objective at unity. Incorporation of the Pareto-Koopmans definition of efficiency occurs in (3). Hence, as was noted in our introduction, the two definitions are related to each other by the duality theory of linear programming.

The latter member of this dual pair of linear programming problems-i.e. (3)-is referred to as the 'envelopment model." This is because, applied successively to each member of a set of $n$ DMUs, it generates an "efficiency frontier'" that 'envelops" a 'production possibility set." The first member of this dual pair-i.e., (2)—is referred to as the 'multiplier model"' because it yields the dual variables that evaluate the outputs and inputs represented by the $x_{i o}$ and $y_{r o}$ for DMUo in a manner that differs from the ordinary uses of weights selected on an a priori basis.

Because it generally has fewer constraints, (3) is easier to compute. Nothing is lost in doing so, moreover, since most computer codes use the simplex algorithm of linear programming which automatically generates solutions to both problems when it is used to solve either of them. Access to the optimal $\mu_{r}$ and $v_{i}$ as well as the optimal $\lambda_{j}$ and $\theta$ is thus provided without extra effort.

Turning now to the non-Archimedean $\varepsilon>0$, we note that it is not a real number and hence cannot be assigned a numerical value. See Ali and Seiford (1990a) who provide examples in which choices of smaller real numbers for the value of $\varepsilon$ lead to worsened results. However, it is not necessary to assign $\varepsilon$ a numerical value. Most DEA computer codes deal with (3) in a two-stage manner as follows: In stage one the slacks in (3) are omitted from the objective. In stage two the sum of the slacks is maximized with $\theta=\theta^{*}$
fixed in the constraints from the solution achieved in stage one.

We might now note that the stage one model is referred to as the "Farrell model" because it is the one used in Farrell (1957). In the economics portion of the DEA literature it is often said to conform to the assumption of "free disposal" because it assigns a zero coefficient to every slack variable as part of the measure of efficiency. In the operations research portion of the DEA literature the "Farrell measure" is said to provide a measure of "weak efficiency" in order to reflect the fact that inefficiencies represented by the non-zero slacks are omitted from this measure.

We conclude this section with the following mathematical formulation of Definition (1).

Definition 2 'Full'" or 'strong'' efficiency-i.,e., Pareto Koopmans efficiency-is attained if and only if an optimal solution to (5) fulfills both of the following conditions:
(i) $\theta^{*}=1$
(ii) All slacks are zero
"Weak" efficiency is attained if only condition (i) is satisfied.

## 4 Measures of distance

The Farrell measure given by (i) in Definition 2 is sometimes referred to as a measure of "distance." See Grosskopf et al. (1999, p. 609), for instance. This is not correct. Dmitruk and Koshevoy (1991) in their critique of Fare and Lovell (1978) say it is not a distance function and, instead, refer to it as a 'gauge function.' However, they fail to say what this means in the way of a measure of efficiency or why it is satisfactory for use in DEA-like evaluations. ${ }^{1}$

To clear up the confusion we note that the "Farrell'" measure is commonly referred to as a "radial measure" but this is not revelatory for its uses in DEA. We say, instead, that the 'Farrell'" measure mathematically represents the ratio of two measures of distance: (a) the distance on a ray from the origin to the point with coordinates that represents the performance of the DMUo being evaluated and (b) the distance from the origin to the point where this ray intersects the frontier. We also show that
$0 \leq \theta^{*}=\frac{d\left(0, x_{1}\right)}{d\left(0, x_{2}\right)} \leq 1$,

[^1]where the denominator refers to the distance from the origin to the point being evaluated, and the numerator refers to distance from the origin to the point of intersection with the frontier, with both points lying on the ray described in (a) and (b).

To show what is mathematically involved, we focus on the $l_{p}$ metric defined by
$\ell_{p}=d\left(x_{1}, x_{2}\right)=\left(\sum_{j=1}^{n}\left|x_{1 j}-x_{2 j}\right|^{1 / p}\right), p \geq 1$.
We focus on this measure because it is, by far, the most commonly used measure as in, for instance, the $\ell_{2}$ metric that is used in least squares statistical estimates. We now note that (5) satisfies all of the conditions for a distance measure in (general) vector spaces-viz.
(i) non-negativity: $d\left(x_{1}, x_{2}\right) \geq 0, d\left(x_{1}, x_{2}\right)=0$ iff $x_{1}=x_{2}$.
(ii) symmetry: $d\left(x_{1}, x_{2}\right)=d\left(x_{2}, x_{1}\right)$.
(iii) triangle axiom: $d\left(x_{1}, x_{3}\right) \leq d\left(x_{1}, x_{2}\right)+d\left(x_{2}, x_{3}\right)$,
where $x_{1}, x_{2}$ and $x_{3}$ are points in an n-dimensional vector space. See Appendix A in Charnes and Cooper (1961). See also Saaty and Braun (1964, p. 369) where distance is mathematically linked to the definitions of a metric space. ${ }^{2}$

We now put this all together with the following,

## Theorem 1 Every $\ell_{p}$ metric, including $\ell_{\infty}$ satisfies (4):

Proof Because the points $x_{1}$ and $x_{2}$ are on the same ray we have $x_{2}=k x_{1}$ for some $k \geq 1$. Using (4) we have $1 /$ $\theta^{*}=k \geq 1$ representing the relative difference of distances from the origin to $x_{1}$ and $x_{2}$, respectively, with $k>1$ representing a measure of inefficiency and $\theta^{*}$ a measure of efficiency, and $\theta^{*}, k=1$ only when $x_{1}=x_{2}$. Using $x_{2}=k x_{1}$ because they are on the same ray,

$$
\begin{aligned}
& \frac{\left(\sum_{j=1}^{n}\left(\left|x_{11}\right|^{p}+\left|x_{12}\right|^{p}+\cdots+\left|x_{1 n}\right|^{p}\right)\right)^{1 / p}}{\left(\sum_{j=1}^{n}\left(\left|x_{21}\right|^{p}+\left|x_{22}\right|^{p}+\cdots+\left|x_{2 n}\right|^{p}\right)\right)^{1 / p}} \\
& \quad=\frac{\left(\sum_{j=1}^{n}\left(\left|x_{11}\right|^{p}+\left|x_{12}\right|^{p}+\cdots+\left|x_{1 n}\right|^{p}\right)\right)^{1 / p}}{\left(\sum_{j=1}^{n}\left(\left|k x_{11}\right|^{p}+\left|k x_{12}\right|^{p}+\cdots+\left|k x_{1 n}\right|^{p}\right)\right)^{1 / p}}=\frac{1}{k}
\end{aligned}
$$

[^2]

Fig. 1 Distance measures
for any $p$. Hence, setting $l / k=\theta^{*}$ we find that (4) is satisfied since $k \geq 1$.

Figure 1, above, illustrates the situation for the case of 2 inputs and one output with the solid lines representing level lines or isoquants with $k \geq 1$ relating the points $x_{1}$ and $x_{2}$ to each other. Hence we have $1 / \theta^{*}=k \geq 1$ representing the relative difference in distance from the origin to $x_{1}$ and $x_{2}$, respectively, with $k$ representing a measure of inefficiency. The two will coincide with $k=\theta^{*}=1$ if and only $x_{1}=x_{2}$ in which case efficiency is achieved.

We also have the following
Corollary The resulting measures are "units invariant." That is, the same values for $k$ and $\theta^{*}$ will be obtained with any unit of measure being employed for $x_{1}$ and $x_{2}$ because the numerator and denominators will be measured in the same units and will therefore cancel.

A weakness of the Farrell measure lies in the fact that it may fail to distinguish between efficient and inefficient performances. This is illustrated by $x_{3}$, an observation which is on the frontier with a Farrell measure of unity. This is only weak efficiency, however, since movement from $x_{3}$ to A reduces input 2 without increasing input 1 . See Definition 1. Further, the Farrell measure for $x_{4}$ is given by $\theta^{*}<1$ with intersection at C where $k>1$. This does not eliminate all of the inefficiencies, however, since movement from $C$ to $B$ reduces input 1 without increasing input 2, as implied by the Pareto-Koopmans definition of inefficiency.

## 5 Additive model

Various models with associated measures have been devised to eliminate this weakness of the Farrell model. An example is the "additive model", which was introduced in

Charnes et al. (1985). This model can be represented by setting $\theta^{*}=1$ in (3). The conditions for full or "strong" efficiency set forth in Definitions 2 then give way to

Definition 3 Full or Strong (Pareto-Koopmans) efficiency is attained for DMUo with an additive model if and only if all slacks are zero in an optimum solution. That is, if and only if $s_{i}^{-^{*}}=s_{r}^{+*}=0 \forall i, r$ in (6), below.

It is to be noted that the objective being optimized in (3) provides measures of two different types of efficiency: (a) $\min \theta=\theta^{*} \leq 1$ is a measure of 'purely" technical efficiency in which a value of $1-\theta^{*}$ provides a measure of the inefficiencies that can be removed without altering any of the input proportions and (b) $\max \sum_{i=1}^{m} s_{i}^{-}+\sum_{r=1}^{s} s_{r}^{+}=$ $\sum_{i=1}^{m} s_{i}^{-*}+\sum_{r=1}^{s} s_{r}^{+*}$ in the second stage for solving (3) maximizes the remaining inefficiencies that represent changes in input proportions (or mixes) that can also be made without altering the already achieved value of $\theta^{*}$. In this way all inefficiencies are accounted for. Hence we say that the measure, $z_{o}^{*}$, in (3) is 'complete" because it accounts for all inefficiencies that the model can identify. See Cooper et al. (1999a, b).

The additive model achieves this same result by eliminating $\theta$ from (3) and replacing the objective with
$\max \left(\sum_{i=1}^{m} s_{i}^{-}+\sum_{r=1}^{s} s_{r}^{+}\right)$.
This uses the $\ell_{1}$ measure of distance as illustrated in Fig. 1 by the horizontal and vertical lines with associated values $s_{1}^{-}$and $s_{2}^{+}$representing the reductions in input 1 and input 2 that are required for $x_{4}$ to achieve efficiency at B . The resulting measure of inefficiency for the performance of $x_{4}$ in this $\ell_{1}$ metric is $s_{1}^{-*}+s_{2}^{+*} \geq 0$.

Like $z_{o}^{*}$ in (3), the additive measure (6) is "complete." That is, as specified in Cooper et al. (1999a, b), this measure reflects all of the inefficiencies that the model can identify. Moreover, it has other attractive properties such as the property of 'translation invariance" when the constraint $\sum_{j=1}^{n} \lambda_{j}=1$ is adjoined so that, inter alia, uses of the additive model are not confined to positive-or even nonnegative—values of the $x_{i j}, y_{r j}$. See Ali and Seiford (1990b).

For clarity, as well as for conformance to common usage, (3) is referred to as the CCR model even when the constraint $\sum_{j=1}^{n} \lambda_{j}=1$ is included. Introduced in Charnes, Cooper and Rhodes (1978) it led to the additive model that was introduced in Charnes, et al. (1985). The latter, in turn, opened a path for the further developments that we describe in the next section. We conclude the present section with the following theorem which relates the additive to the CCR model.

Theorem 2 The additive model will identify a DMU as efficient if and only if the CCR model represented in (3) also identifies it as efficient.

This theorem is proved in Ahn et al. (1988) so we need not prove it here. We do note, however, that an immediate corollary is that achievement of efficiency with the additive model is sufficient but not necessary for the achievement of efficiency in the Farrell model.

Remark T. Coelli (1998) has noted the possibility of "far removed" reference sets to conduct evaluations with the additive model. This means that the additive model is likely to be unsatisfactory for identifying benchmark candidates to improve the performances of inefficient DMUs. See Coelli (1998) for a discussion of how to choose a best reference set of DMUs.

## 6 Alternative models and measures

Various attempts have been made to address deficiencies in the additive model. The "Slacks-Based Measure" (SBM) of Tone (2001) is an example in which the property of units invariance is preserved. We present this model in vectormatrix form as follows:

$$
\min \rho=\frac{1-\frac{1}{m} \sum_{i=1}^{m} s_{i}^{-} / x_{i o}}{1+\frac{1}{s} \sum_{r=1}^{s} s_{r}^{+} / y_{r o}}
$$

subject to

$$
\begin{aligned}
x_{o} & =X \lambda+s^{-} \\
y_{o} & =Y \lambda-s^{+} \\
o & \leq \lambda, s^{-}, s^{+} .
\end{aligned}
$$

These constraints are the same as for the additive model. The objective, however, is replaced by representing the input and output slacks in fractional programming form. The slacks are used to account for all inefficiencies in a manner that differs from the additive model and yields a measure that satisfies
$0 \leq \min \rho=\rho^{*} \leq 1$,
with $\rho^{*}=1$ if and only if DMU is strongly (i.e., ParetoKoopmans) efficient. This efficiency is attained if and only if all slacks are zero just as for the additive model. See Definition 3.

For the Charnes-Cooper transformation, $\mathfrak{t}$, Tone uses $t\left(1+\frac{1}{s} \sum_{r=1}^{s} s_{r}^{+} / y_{r o}\right)=1$ so
$t=1 /\left(1+\frac{1}{s} \sum_{r=1}^{s} s_{r}^{+} / y_{r o}\right)>0$.
Multiplying numerator and denominator in the objective of (7) by this $t>0$ leaves its value unchanged. We can therefore replace (7) by
$\min \rho \prime=t-\frac{1}{m} \sum_{i=1}^{m} s_{i}^{-\prime} / x_{i o}$
subject to

$$
\begin{align*}
\mathrm{t} x_{i o} & =\sum_{j=1}^{m} x_{i j} \lambda_{j}^{\prime}+s_{i}^{-\prime}, i=1, \ldots, n \\
\mathrm{t}_{r o} & =\sum_{j=1}^{n} y_{r j} \lambda_{j}^{\prime}-s_{r}^{+\prime}, r=1, \ldots, s  \tag{8}\\
1 & =t+\frac{1}{s} \sum_{r=1}^{s} s_{r}^{+\prime} \\
0 & \leq \lambda_{j}^{\prime}, s_{i}^{-\prime}, s_{r}^{+\prime}, \forall i, j, r
\end{align*}
$$

where $s_{i}^{\prime^{\prime}}=t s_{i}^{-}, s_{r}^{+^{\prime}}=t s_{r}^{+}, \lambda_{j}^{\prime}=t \lambda_{j}, \forall i, j, r$.
Full (or strong) Pareto-Koopmans efficiency is achieved if and only if $s_{i}^{-^{*}}=0 \forall i$ and $t^{*}=1$. The latter implies $s_{r}^{+^{+}} *$ $=0 \forall r$, via (7.2), so $t^{*}$ plays a role analogous to the transformation used in going from (1) to (2) but with the additional property that $t^{*}=1$ implies $s_{r}^{+^{\prime *}}=0, \forall r$. Evidently the objective in (7) and hence in (8) is units invariant and $s_{i}^{-^{\prime}} \leq x_{i o} \forall i \quad s_{i}^{-^{\prime}} / x_{i o} \leq 1, \forall i$, so $\rho^{\prime *}$, is stated in terms of the average proportion of input inefficiency.

Theorem 4 SBM, as in (8), will identify a DMUo as efficient if and only if the additive model also identifies it as efficient.

Proof This is obvious.
Bardhan et al. (1996) show other routes which, inter alia, can also be used to relate (8) and the additive model to other models such as the FDH (=Free Disposal Hull) model of Tulkens et al. (1993). They also develop ways in which this measure can be made more attractive. However, we do not follow their developments. Instead, we turn to another model (and measure) and show this that it, too, is related to the additive model. This is the ''Russell Measure," and model, introduced in Färe and Lovell (1978).

The formulation in Färe and Lovell (1978) uses only the inputs in their measure of efficiency. This measure is incomplete. We therefore turn to the formulation used by Färe, Grosskopf of and Lovell $(1985,1994)$,
$\min R\left(\sum_{i=1}^{m} \theta_{i}, \sum_{r=1}^{s} 1 / \phi_{r}\right)=\frac{\sum_{i=1}^{m} \theta_{i}+\sum_{r=1}^{s} 1 / \phi_{r}}{\mathrm{~m}+s}$ subject to
$\theta_{i} x_{i o} \geq \sum_{j=1}^{n} x_{i j} \lambda_{j}$,
$\phi_{r} y_{r o}=\sum_{j=1}^{n} y_{r j} \lambda_{j}, \quad r=1, \ldots, s$
$0 \leq \theta_{i} \leq 1,1 \leq \phi_{r}, \quad \forall i, r$
$0 \leq \lambda_{j} \quad j=1 \ldots, n$.
The objective is therefore to minimize the average value of the input plus output efficiencies as measured by the optimal $\theta_{i}$ and $1 / \phi_{r}$. The resulting measure is complete as well as units invariant and has other attractive properties. However, it is computationally difficult, ${ }^{3}$ so we turn to alternatives. Färe et al. (1985, 1994), for example, decompose the Russell measure into input and output oriented variants. This loses the property of completeness, however, so we turn to another version which has recently become available in Pastor et al. (1999). See also Bardhan et al. (1996).

This model takes the following 'fractional programming' form
$\min R\left(x_{o}, y_{o}\right)=\frac{\sum_{i=1}^{m} \theta_{i} / m}{\sum_{r=1}^{s} \phi_{r} / s}$
subject to
$\theta_{i} x_{i o} \geq \sum_{j=1}^{n} x_{i j} \lambda_{j}, \quad i=1, \ldots, m$
$\phi_{r} y_{r o} \geq \sum_{j=1}^{n} y_{r j} \lambda_{j}, \quad r=1, \ldots, s$
$0 \leq \theta_{i} \leq 1,1 \leq \phi_{r}, \quad \forall i, r$
$0 \leq \lambda_{j}, \quad j=1 \ldots, n$.
Pastor et al. refer to this as the "Enhanced Russell Graph Measure of Efficiency', but we shall refer to it as the ERM (Enhanced Russell Measure) model . The objective in (10) gives

[^3]$0 \leq \frac{\sum_{i=1}^{m} \theta_{i} / m}{\sum_{r=1}^{s} \phi_{r} / s}=\left(\sum_{i=1}^{m} \theta_{i} / m\right)\left(\sum_{r=1}^{s} \phi_{r} / s\right)^{-1} \leq 1$,
which means that it can be interpreted as the product of the average input and average output efficiencies with the geometric mean (=square root) of the optimal value of (11) serving as the measure of efficiency. Alternatively the numerator and denominator values, which are also available, can be separately interpreted as the average input efficiency and the average output inefficiency, respectively, with $R\left(x_{o}\right.$, $\left.y_{o}\right)=1$ if and only if full efficiency is attained. Thus full efficiency is attained if and only if, at an optimum,
$\theta_{i}^{*}=\phi_{r}^{*}=1$, all $i$ all $r$.
This model has the properties of completeness and units invariance. It also has other attractive properties such as "strong [=strict] monotonicity," etc., as described in Pastor et al. (1999). As yet another property, it can be related to SBM and thus to the additive model. In fact, we have the following theorem which we now prove.

Theorem 5 ERM and SBM are equivalent in that $\lambda_{\mathrm{j}}^{*}$ values that are optimal for one are also optimal for the other.

Proof A necessary condition for optimality in ERM is that the constraints must be satisfied as equalities. We can therefore replace the above constraints by
$\theta_{i}=\sum_{j=1}^{n} x_{i j} \lambda_{j} / x_{i o}, i=1, \ldots, m$
$\phi_{r}=\sum_{j=1}^{n} y_{r j} \lambda_{j} / y_{r o}, r=1, \ldots, s$.
Following Pastor et al. (1999) or Bardhan et al. (1996) we next set
$\theta_{i}=\frac{x_{i o}-s_{i}^{-}}{x_{i o}}=1-\frac{s_{i}^{-}}{x_{i o}}, i=1, \ldots, m$
$\phi_{r}=\frac{y_{r o}+s_{r}^{+}}{y_{r o}}=1+\frac{s_{r}^{+}}{y_{r o}}, r=1, \ldots, s$.
Substituting these values into the immediately preceding expression, we get
$x_{i o}=\sum_{j=1}^{n} x_{i j} \lambda_{j}+s_{i}^{-}, i=1, \ldots, m$
$y_{r o}=\sum_{j=1}^{n} y_{r j} \lambda_{j}-s_{r}^{+}, r=1, \ldots, s$

These are the same as the 'additive model' constraints used in SBM.

Turning to the additional conditions $0 \leq \theta_{i} \leq 1,1 \leq \phi_{r}$ in ERM we again utilize (13) to obtain $0 \leq s_{i}^{-} \leq x_{i o}$ and $0 \leq s_{r}^{+} \forall i$, $r$. The condition $s_{i}^{-} \leq x_{i o}$ is redundant since, as noted earlier, it is satisfied by the constraints of the additive model and may therefore be discarded. We then have the usual non-negativity conditions satisfied as $0 \leq s_{i}^{-}, 0 \leq s_{r}^{+}, \forall i, r$.

Turning to the objective in (10) this same substitution from (13) yields
$\frac{s}{m} \frac{\sum_{i=1}^{m} \theta_{i}}{\sum_{r=1}^{s} \phi_{r}}=\frac{s}{m} \frac{\sum_{i=1}^{m}\left(1-\frac{s_{i}^{-}}{x_{i o}}\right)}{\sum_{r=1}^{s}\left(1+\frac{s_{r}^{+}}{y_{r o}}\right)}=\frac{1-\frac{1}{m} \sum_{i=1}^{m} \frac{s_{i}^{-}}{x_{i o}}}{1+\frac{1}{s} \sum_{r=1}^{s} \frac{s_{r}^{+}}{y_{r o}}}$
which is the same as the objective of SBM in (7). We may therefore use SBM to solve ERM, or vice versa, with assurance that any $\lambda_{j}^{*}$ that are optimal for one model are also optimal for the other.

ERM like SBM does not possess the property of translation invariance. See Portela et al. (2003) for an example of the importance of the property of translation invariance. Hence their uses are confined to positive ranges in the data. We therefore turn to another model that is both units invariant and translation invariant. This is the RAM (Range Adjusted Measure) model introduced in Cooper et al. (1999a, b), which takes the following form,
$\max \frac{1}{m+s}\left(\sum_{i=1}^{m} s_{i}^{-} / R_{i}^{-}+\sum_{r=1}^{s} s_{r}^{+} / R_{r}^{+}\right)$
subject to
$x_{i o}=\sum_{j=1}^{n} x_{i j} \lambda_{j}+s_{i}^{-}, i=1, \ldots, m$
$y_{r o}=\sum_{j=1}^{n} y_{r j} \lambda_{j}-s_{r}^{+}, r=1,2 \ldots s$
$1=\sum_{j=1}^{n} \lambda_{j}$
$0 \leq \lambda_{j}, s_{i}^{-}, s_{r}^{+}, \forall i, j, r$.
Here $R_{i}^{-}$and $R_{r}^{+}$represent the "ranges," and a measure of efficiency is given by
$0 \leq \Gamma=1-\frac{1}{m+s}\left(\sum_{i=1}^{m} \frac{s_{i}^{-}}{R_{i}^{-}}+\sum_{r=1}^{s} \frac{s_{r}^{+}}{R_{r}^{+}}\right) \leq 1$,
which, as can be seen, is a "weighted" version of the additive model.

Reciprocals of these ranges provide data dependent weights and are not determined in an a priori manner. They are also not affected by the addition of an arbitrary constant in any input or output row since

$$
\begin{gather*}
R_{i}^{-}=\bar{x}_{i}-\underline{x}_{i}=\left(\bar{x}_{i}+d_{i}\right)-\left(\underline{x}_{i}+d_{i}\right), i=1, \ldots, n \\
R_{r}^{+}=\bar{y}_{r}-\underline{y}_{r}=\left(\bar{y}_{r}+c_{r}\right)-\left(\underline{y}_{r}+c_{r}\right), r=1, \ldots, s . \tag{16}
\end{gather*}
$$

where $\bar{x}_{i}, \underline{x_{i}}$ and $\bar{y}_{r}, \underline{y_{r}}$ are maximal and minimal observations and the $d_{i}$ and $\bar{c}_{r}$ are arbitrary constants. The condition $\sum_{j=1}^{n} \lambda_{j}=1$ accords the same property to the constraints since, as shown in Ali and Seiford (1990b), the adjunction of this convexity condition makes all additive models translation invariant. This model is therefore both units and translation invariant. This makes it possible to treat nonpositive inputs or outputs by converting them to positive values. See the exchange between Steinmann and Zweifel (2001) and Cooper et al. (2001). This convexity condition also makes it possible to handle situations in which the ranges are zero by simply omitting the slack variables from the objective and the constraints with which they are associated since these slacks will necessarily be zero.

We conclude this section with the following.
Theorem 6 The measure in the RAM model is affine invariant. ${ }^{4}$ That is, it is invariant to transformations of the form $x_{i j}^{\prime}=b_{i}\left(x_{i j}+a_{i}\right), y_{r j}^{\prime}=c_{r}\left(y_{r j}+d_{r}\right)$ for each $i$ and each $r$ for all $j=1, \ldots, n . a_{i}, b_{i}, c_{r}, d_{r}$ are constant with $b_{i}, c_{r}$ $>0$ but the constants are otherwise arbitrary.

This theorem is proved in Cooper et al. (2006) so we do not prove it here. Instead we note that all of these models, including RAM, represent extensions of the additive model. As additional evidence of its fundamental character of that model we might note that the additive model provides a bridge for contacts with other disciplines such as are represented by multiple objective decision making in operations research and LAV (Least Absolute Value) estimation in statistics. See Cooper (2005) and Charnes et al. (1985).

## 7 Conclusions and prospects for the future

We have now covered some of the measures and models that are available from past accomplishments. We do not go further into these models and other variants that have been derived to deal with non-discretionary variables and/ or returns to scale, etc. We have also confined attention to 'technical efficiency" and so have omitted important developments in topics like 'total," 'allocative'" and "returns-to-scale" efficiencies. Even in the treatment of technical efficiency we did not include attractive properties of the "multiplicative model" other than to note that it is unique in being able to provide estimates of "exact'" (i.e., numerically valued) scale elasticities without any need to use partial derivatives or like mathematical concepts. See Banker et al. (2004). See also Charnes et al. (1983).

[^4]The topics we have covered are sufficient, however, to show the great progress that has been made (and is being made) in DEA as well as problems that merit attention in future research. This includes the possibility of combining desirable benchmarking selections with other desirable properties such as "completeness," "units invariance" and "translation invariance" such as were illustrated for the RAM model described in the last part of the preceding section.

This is not the end of the road. As already noted, Coelli (1998) has opened an avenue that invites exploration in the form of algorithms directed to identifying better reference groups-e.g., for benchmarking-without losing properties that are also desirable. See also Portela et al. (2004) for a survey of alternatives and a proposed use of FDH models and methods. Additional possibilities are also open for exploitation that include the use of DEA network computing capabilities like those described in Sueyoshi and Honma (2003). In addition to providing an ability to deal with very large problems more economically and in smaller times, this path also opens the possibility of combining results from several models to obtain properties that are not available from any one of them when used separately. It goes without saying that such increases in computational power and efficiency will also be needed as extensions are made to dynamic DEA models in which the number of DMUs is likely to increase in a combinatorial manner. See Sengupta (1995) and Färe and Grosskopf (1996). Finally, opportunities exist for joint uses of now available methods on the same model and data that make it possible to identify sources of trouble in the data which can be corrected by use of a frontier (in contrast to an OLS regression). See Brockett et al. (2007) for a use of DEA to locate defects in the data and how they can be dealt with that range from sample bias to errors in processing the data.

As noted in the preceding section of this paper, the RAM model deals with data dependent weights in the form of ranges (or their reciprocals) that makes this model affine invariant. This is not the only such possibility. Lovell and Pastor (1995), for example, utilize standard deviations, or their reciprocals, instead of ranges. This choice, however, is accompanied by limitations like those described on p. 21 of Cooper et al. (1999a, b). Also data dependent weights are not the only possible choice so other avenues need to be explored. For guidance in such explorations we recommend the discussion of "goal vectors" on pp. 120-125 in Thrall (1996) who brings to bear his extensive knowledge of "dimensional analysis," as employed in mathematics and the natural sciences, to help in making selections.

Another inviting avenue for development lies in the selection of objectives. Here we have restricted consideration to "strong" versus "weak" efficiency. Other choices could include the attainment of "satisfactory" levels of
efficiency such as are comprehended in the concept of "satisficing" due to Simon (1957), Chap. 14,-a concept that has been accorded widespread acceptance in cognitive psychology. See Gigerenzer (2004). Here, too, a start for uses of "satisficing" objectives in DEA has been made by according it a chance-constrained programming formulation in Cooper et al. (1996). Transformed to equivalent deterministic forms this kind of objective has also been useful in locating organizational (or budgetary) slack in U.S. Air Force activities. See Bowlin (1984) and Charnes et al. (1989).

Still another course of development along these lines takes the form of expanding DEA beyond efficiency evaluations. One such course would expand DEA from "efficiency" to "effectiveness" evaluations. See p. 66 in Cooper et al. (2000) or p. 63 in Cooper et al. (2006) for definitions of these two concepts and their relations to each other. Once again, a start has been made in Prieto and Zofio (2001) who use DEA to evaluate the effectiveness of Spanish municipalities in achieving goals (specified by the central government) in areas such as the provision of water quality, sewage treatment, paving and lighting and artistic and sporting facilities.

Continuing on this path one can envision mixtures such as in Golany and Thore (1997) who evaluate the social performances of countries. Here the "inputs" and "outputs" consist of mixtures such as "infant mortality," a social performance factor, and gross domestic product per capita, an economic performance factor. See also the study by Takamura and Tone (2003) which is devoted to relocating the political capital of Japan from Tokyo to other locations (which is now under consideration by the Japanese Diet) and considers factors like "safety" in the event of an earthquake or a volcano, along with reductions of congestion in Tokyo and increases in congestion in other locations-which are compared for "best" and "worst" possibilities.

We might also consider uses of DEA to synthesize "social indicators" such as is done by Ramanathan (2007) who evaluates the social performances of Middle East and North African Arab countries with "inputs" like "\% infant mortality" and "outputs" like "\% female teachers" and "life expectancy at birth." To determine inputs vs. outputs, Ramanathan uses the conceptual power of the fractional objective in (1) to determine whether an increase in the designated attribute should be placed in the numerator or the denominator-and hence should be regarded as an "input" or an "output"-according to whether an increase in the designated attribute would increase or decrease the value of the DEA ratio score for each of 15 Arab countries. See also Despotis (2005) who adapts a similar approach and uses DEA to improve upon the "Human Development Index" used by the United Nations in
addition to its "Standard of Living Index" for different countries.

Of course, there are advances still to be made in other areas such as the joining of statistical considerations to DEA estimates. See, e.g., the use of maximum likelihood estimates in Banker and Natarajan (2004) or the use of bootstrapping in Simar and Wilson (2004). The paths we have already outlined should both profit from and contribute to such statistical developments. Here, too, additional alternatives invite exploration. One example is the use of DEA in combination with ordinary least squares which is accomplished in the following two-stage manner: DEA is employed in stage one to identify the observations associated with efficient performances. In stage two the first-stage results are incorporated in the form of "dummy variables'" in ordinary least squares regressions to obtain a new approach to estimating stochastic frontiers. In an extended simulation this approach was found to give better results than ordinary least squares or stochastic frontier regressions. See Bardhan et al. (1998). This two-stage approach also proved to be the best of three alternative approaches in an actual application to evaluate the efficiency of alternative advertising strategies for use in military recruitment. See Brockett et al. (2002, 2004, 2007).

Another approach to treating imprecise data pioneered by Cook et al. (1993) is directed to the treatment of ordinal data. This was extended by Cooper et al. (2001) to include the treatment of bounds on the data and also on the variables, as in the Assurance Region approach of Thompson et al. (1986, 1990), which is applied to evaluating the branch office performances of a Korean mobile telecommunication company to allow for regional differences such as hills or mountains that affect performances. See Cooper et al. (2001). Work along these lines continues in the form of simplifying the transformations used to convert these nonlinear problems in ways that cast additional light on the performances of Assurance Regions and like approaches in DEA. See Park (2004).

Combining these imprecise data treatments with the statistical approaches discussed in Banker and Natarajan (2004) or Simar and Wilson (2004) should greatly extend the power and the applicability of both. Their combination should also extend the range of applications for DEA and thereby expand this source of problems and opportunities for further development in DEA. This could include helping DEA to expand into the area of consumer behavior, an area which has only begun to be studied along lines like those that are suggested by Lancaster (1966). See Lee et al. (2005).

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[^1]:    ${ }^{1}$ See Sobczyk (1956) for a definition of gauge functions and their uses with convex regions.

[^2]:    ${ }^{2}$ The concept of a "directed distance" is discussed in terms of the distance from a point to a hyperplane on page 164 in Charnes and Cooper (1961).

[^3]:    ${ }^{3}$ Our attention has recently been called to Sueyoshi and Sekitani (2007) that uses cone programming based approach to solve (9) in a straightforward manner. Hence practically implementable alternative are now available. See Cooper et al. (2007) for the advantages of using ERM rather than RM to deal with the aggregation problem in DEA.

[^4]:    ${ }^{4}$ Russell (1985) refers to this property as "commensurability' but we think that "affine invariant"' is more descriptive.

