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Sensitivity and Stability of the Classifications of Returns to Scale in Data Envelopment Analysis

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Abstract

Sensitivity of the returns to scale (RTS) classifications in data envelopment analysis is studied by means of linear programming problems. The stability region for an observation preserving its current RTS classification (constant, increasing or decreasing returns to scale) can be easily investigated by the optimal values to a set of particular DEA-type formulations. Necessary and sufficient conditions are determined for preserving the RTS classifications when input or output data perturbations are non-proportional. It is shown that the sensitivity analysis method under proportional data perturbations can also be used to estimate the RTS classifications and discover the identical RTS regions yielded by the input-based and the output-based DEA methods. Thus, our approach provides information on both the RTS classifications and the stability of the classifications. This sensitivity analysis method can easily be applied via existing DEA codes.

Keywords: Data envelopment analysis (DEA), returns to scale (RTS), sensitivity, stability.

1. Introduction

In their seminal paper, Charnes, Cooper and Rhodes (CCR, 1978) coined the term data envelopment analysis (DEA) to describe a new methodology for estimating the relative efficiencies and inefficiencies of decision making units (DMUs). One research issue which has received widespread attention in the rapidly growing field of DEA is the characterization of returns to scale (RTS).

Seiford and Zhu (1999) establish the equivalence of the following three methods for characterizing RTS which have appeared in the literature. (See Golany and Yu (1997) and Tone (1996) for additional discussion.)

Banker (1984) introduced the CCR RTS method using the sum of the intensity variables in the CCR model to indicate RTS. Banker, Charnes and Cooper (BCC, 1984) developed an alternative approach using the free variable in the BCC dual model. These two basic RTS methods have been modified to deal with situations where DEA formulations have multiple optimal solutions (Banker and Thrall (1992) and Zhu and Shen (1995)). Finally, Färe, Grosskopf and Lovell (1985, 1994) proposed a rather natural RTS approach by using scale efficiency. Their method exploits the natural nesting of the three RTS frontiers that exhibit constant, nonincreasing and variable returns to scale (CRS, NIRS and VRS).

While the sensitivity analysis of efficiency classifications in DEA has been extensively studied (Seiford, (1994, 1996, 1997)), the issue of robustness of RTS estimation and classification appears to have been ignored. This is surprising since RTS classifications provide important information for improving an individual DMU's performance when scale inefficiencies are detected. Furthermore since RTS estimates in DEA only hold locally, it is important to investigate the stability of the RTS classifications.

The current paper addresses the sensitivity of RTS classifications in DEA. Since the three existing RTS methods are equivalent (Seiford and Zhu, (1999), we utilize the CCR RTS method, based upon the sum of the optimal lambda values in the CCR model, to address the sensitivity issue in RTS estimation.

We develop several linear programming formulations for investigating the stability of RTS classifications. The possible data perturbations for preserving the DMUs' RTS classifications—constant, increasing or decreasing returns to scale (CRS, IRS or DRS) are computed from the optimal values.

A by-product of our RTS sensitivity analysis measure is an alternative method for characterizing RTS. It is easily seen that the optimal values to the newly developed linear programming problem can be used to identify the RTS classification. This new RTS method requires information on the optimal basis set from the CCR model. The newly developed measures yield information on both the RTS classifications and the stability of these RTS classifications by solving two DEA-type formulations.

Note that the input-based and the output-based CCR models may produce different RTS classifications. Therefore the sensitivity issue is addressed for the RTS results obtained respectively from the two versions of CCR models. Nevertheless, note also that the two CCR models do yield some identical RTS regions (see Seiford and Zhu, (1999). Our new measures also can be used to discover these identical RTS regions.

The remainder of this paper is organized as follows. Section 2 discusses the basic DEA models and the CCR RTS method. Section 3 develops the sensitivity analysis method for the RTS estimation when the summation of lambda variables is always equal to one for the CRS DMUs in all possible optimal solutions to the CCR model, and the CCR efficient facets satisfy convexity. We examine the sensitivity issue under both the input-oriented and the output oriented CCR RTS methods. Section 4 addresses RTS stability in the general situation of no regularity conditions. Conclusions are given in section 5. Simple numerical examples and a figure which illustrate the input oriented method of section 3 and the case of multiple optimal lambda summations are provided in appendices A and B. Appendix C applies the method to a real world data set.

2. Preliminaries

Suppose we have a set (***J***) of DMUs. Each DMU_j ($j \in J$), produces an amount y_{rj} ($r = 1, 2, \ldots, s$) of s different outputs utilizing amounts x_{ij} ($i = 1, 2, \ldots, m$) of m different inputs. In DEA, the CCR model evaluates the relative efficiency of a specific DMU_o, $o \in J$, with respect to a set of CCR-frontier DMUs (belonging to sets E, E′ or F of Charnes, Cooper and Thrall (1991)) defined $E_o = \{j \mid \lambda_j > 0$ for some optimal solutions for $DMU_o\}$.[1]

$$\begin{aligned} &\min \theta \\ s.t. &\sum_{j\in E_o} \lambda_j x_{ij} \le \theta x_{io} \quad i = 1, 2, \ldots, m; \\ &\sum_{j\in E_o} \lambda_j y_{rj} \ge y_{ro} \quad r = 1, 2, \ldots, s; \\ &\lambda_j \ge 0 \quad j \in E_o \end{aligned} \tag{1}$$

Similarly we can write an output-oriented CCR model

$$\begin{aligned} &\max \vartheta \\ s.t. &\sum_{j\in E_o} \lambda_j x_{ij} \le x_{io} \quad i = 1, 2, \ldots, m; \\ &\sum_{j\in E_o} \lambda_j y_{rj} \ge \vartheta y_{ro} \quad r = 1, 2, \ldots, s; \\ &\lambda_j \ge 0 \quad j \in E_o \end{aligned} \tag{2}$$

If $E_o = J$, then (1) is the original form of the input-oriented CCR model (see Charnes et al. (1994) or Lewin and Seiford (1997) for details). The DMU_j ($j \in E_o$) are called CCR-efficient and form a specific CCR-efficient facet. These DMU_j ($j \in E_o$) appear in optimal solutions where $\lambda_j > 0$.

We can write the CCR model in form of (1) or (2) due to the fact that $\lambda_j = 0$ for all $j \notin E_o$ in the original CCR model when evaluating DMU_o.

Then on the basis of all optimal lambda solutions to (1) (or (2)), the CCR RTS method can be expressed as (Banker and Thrall, (1992)): [2]

The RTS classification for DMU_o is identified as CRS *if and only if* $\sum_{j\in E_o} \lambda_j^* = 1$ in *some* optima, IRS *if and only if* $\sum_{j\in E_o} \lambda_j^* < 1$ in *all* optima, and DRS *if and only if* $\sum_{j\in E_o} \lambda_j^* > 1$ in *all* optima.

LEMMA 1 *For a DMU_o, if we have $\lambda_j^{*(1)}$ ($j \in E_o$) with $\sum_{j\in E_o} \lambda_j^{*(1)} < 1$ and $\lambda_j^{*(2)}$ ($j \in E_o$) with $\sum_{j\in E_o} \lambda_j^{*(2)} > 1$ in (1) (or (2)), then we must have λ_j^* ($j \in E_o$) with $\sum_{j\in E_o} \lambda_j^* = 1$, where (*) represents optimal value.*

Proof. Let $\sum_{j\in E_o} \lambda_j^{*(1)} = d_1$, and $\sum_{j\in E_o} \lambda_j^{*(2)} = d_2$. Define $d = \frac{1-d_1}{d_2-d_1}$. Obviously $0 < d < 1$ and $(1-d)d_1 + dd_2 = 1$.

Let $\lambda_j^* = (1-d)\lambda_j^{*(1)} + d\lambda_j^{*(2)}$ ($j \in E_o$). Then $\sum_{j\in E_o} \lambda_j^* = 1$ and

$$\sum_{j\in E_o} \lambda_j^* x_{ij} \le \sum_{j\in E_o} \left[(1-d)\lambda_j^{*(1)} + d\lambda_j^{*(2)}\right] x_{ij} \le \theta^* x_{io}$$

$$\sum_{j\in E_o} \lambda_j^* y_{rj} = \sum_{j\in E_o} \left[(1-d)\lambda_j^{*(1)} + d\lambda_j^{*(2)}\right] y_{rj} \ge (1-d)y_{ro} + dy_{ro} = y_{ro}$$

Thus λ_j^* ($j \in E_o$) with $\sum_{j\in E_o} \lambda_j^* = 1$ is an optimal solution. ■

Remark 1. This lemma indicates that multiple optimal solutions with $\sum_{j\in E_o} \lambda_j^{*(1)} < 1$ and $\sum_{j\in E_o} \lambda_j^{*(2)} > 1$ are only possible for CRS DMUs.

By the relationship between solutions to the output-based and the input-based CCR models Seiford and Thrall, (1990), we have

LEMMA 2 *Suppose λ_j^* $(j \in E_o)$ and θ^* is an optimal solution to (1). There exists a corresponding optimal solution $\tilde{\lambda}_j^*$ $(j \in E_o)$ and ϑ^* to (2) such that $\tilde{\lambda}_j^* = \frac{\lambda_j^*}{\theta^*}$ and $\vartheta^* = \frac{1}{\vartheta^*}$, or equivalently, $\lambda_j^* = \frac{\tilde{\lambda}_j^*}{\vartheta^*}$ and $\vartheta^* = \frac{1}{\theta^*}$.*

Note that a change in input levels for DMU_o in (1) or a change of output levels in (2) does not alter the RTS nature of DMU_o unless it is moved onto the CCR-efficient frontier. Therefore we limit our investigation to the effect of output changes under (1) and the effect of input changes under (2) on the RTS classification for DMU_o.

Note also that CCR-efficient DMUs continue to exhibit CRS if they are still efficient after data variations. Therefore we may use the sensitivity analysis procedure for the robustness of efficient DMUs in Zhu (1996) and Seiford and Zhu (1998a, 1998b) to investigate the stability of RTS estimation on CCR-efficient DMUs. Hence we only address the sensitivity of RTS classifications for CCR-inefficient DMUs.

3. Sensitivity of the RTS Classification

From (1) and (2), we know that the robustness of the RTS estimate is relative to the CCR-efficient DMUs and DMU_o itself, and is not affected by the other DMUs. We suppose that the CCR-efficient DMUs, i.e., DMU_j $(j \in E_o)$, are fixed and consider the movement of DMU_o.

Note that the different orientations of (1) and (2) may yield different RTS results for DMU_o. Therefore in the development to follow, we discuss the RTS sensitivity issue under (1) and (2) respectively.

(3.A) Sensitivity of the RTS Classifications in Terms of (1)

Note that under (1), if DMU_o exhibits IRS, then decreases in outputs can not change its IRS nature. Likewise, if DMU_o exhibits DRS, increases in outputs can not change its DRS nature unless DMU_o reaches the CCR frontier. Therefore we only consider output increases and decreases respectively for IRS and DRS DMUs.

Since the estimation of RTS in DEA usually considers the proportional change (increase or decrease) in all the outputs of DMU_o achieved by a proportional change in all its inputs, we consider proportional (radial) perturbations for all the outputs of DMU_o. Denote the proportional increase by $\alpha \geq 1$ and the proportional decrease by $\beta \leq 1$, i.e., DMU_o may increase or decrease its outputs respectively by α and β up to αy_{ro} and βy_{ro} $(r = 1, 2, \ldots, s)$ and the RTS classification remains the same.

In order to calculate the values of α and β, we first define the set T_o for DMU_j $(j \in E_o)$:

$$T_o = \left\{ (x, y): \quad \sum_{j \in E_o} \lambda_j x_{ij} \le x_i, i = 1, 2, \ldots, m; \quad \sum_{j \in E_o} \lambda_j y_{rj} \ge y_r, r = 1, 2, \ldots, s; \quad \sum_{j \in E_o} \lambda_j = 1; \lambda_j \ge 0, j \in E_o \right\}$$

Relative to this set, we can now define the following measure:

$$\varphi_o^* = \max\{\varphi_o\colon (x_o, \varphi_o y_o) \in T_o\} \tag{3}$$

where (x_o, y_o) represent the input and output vector of DMU_o and φ_o^* can be calculated as the solution to the linear programming problem:

$$\begin{aligned} &\varphi_o^* = \max \varphi_o \\ s.t. \ &\sum_{j \in E_o} \lambda_j x_{ij} \le x_{io} \qquad i = 1, 2, \ldots, m; \\ &\sum_{j \in E_o} \lambda_j y_{rj} \ge \varphi_o y_{ro} \quad r = 1, 2, \ldots, s; \\ &\sum_{j \in E_o} \lambda_j = 1 \\ &\lambda_j \ge 0 \qquad\qquad j \in E_o \end{aligned}$$

The above formulation is similar to the output-based BCC model but the reference set is restricted to the CCR-efficient DMUs. Four possible cases are associated with (3), that is, $\varphi_o^* = 1$, $\varphi_o^* < 1$, $\varphi_o^* > 1$ or (3) is infeasible.

LEMMA 3 *If DMU_o exhibits DRS, then (3) is feasible.*

Proof. We introduce the new variables:

$$\begin{aligned} \text{Let} \hat{\varphi}_o &= \hat{\theta}\varphi_o = 1 \text{ so } \hat{\theta} = \varphi_o^{-1} > 0 \\ \hat{\lambda}_j &= \hat{\theta}\lambda_j = \varphi_o^{-1}\lambda_j \ (j \in E_o) \end{aligned}$$

Thus multiplying all constraints by $\hat{\theta}$ in (3) gives

$$\begin{aligned} &\min \hat{\theta} \\ s.t. \ &\sum_{j \in E_o} \hat{\lambda}_j x_{ij} \le \hat{\theta} x_{io} \qquad i = 1, 2, \ldots, m; \\ &\sum_{j \in E_o} \hat{\lambda}_j y_{rj} \ge y_{ro} \qquad r = 1, 2, \ldots, s; \\ &\sum_{j \in E_o} \hat{\lambda}_j = \sum_{j \in E_o} \lambda_j \varphi_o^{-1} = \varphi_o^{-1} = \hat{\theta} \\ &\hat{\lambda}_j, \lambda_j \ge 0 \qquad\qquad j \in E_o \end{aligned} \tag{4}$$

Since DMU_o exhibits DRS, then $\sum_{j \in E_o} \lambda_j^* > 1$ in (1). Let $\sum_{j \in E_o} \lambda_j^* = \tilde{\theta}$. Obviously, $\tilde{\theta} > \theta$ is a feasible solution to (1). Therefore λ_j^* $(j \in E_o)$ and $\tilde{\theta}$ are also a feasible solution to (4). Therefore (3) is feasible. ■

From lemma 1 we know that if the following regularity condition is true, then RTS classifications can be uniquely determined by $\sum_{j \in E_o} \lambda_j^*$ in any optimal solution to (1) (or (2)).

Regularity Condition (RC1). $\sum_{j \in E_o} \lambda_j^* = 1$ in all possible optimal solutions for the CRS DMUs.

Note that multiple optimal solutions of lambda variables may occur even under RC1. We also require the following regularity condition (RC2) on the convexity of the CCR efficient facet. RC2 is closely related to the concept of "face regularity" of Thrall (1996).

Regularity Condition (RC2). Suppose E_o forms an efficient facet. Then, any convex combination of CCR efficient DMUs in E_o is still on the same efficient facet.

THEOREM 1 *Suppose regularity conditions RC1 and RC2 hold. Then*

(a) CRS prevail for DMU_o if and only if $\varphi_o^* = 1$;

(b) DRS prevail for DMU_o if and only if $\varphi_o^* < 1$;

(c) IRS prevail for DMU_o if and only if $\varphi_o^* > 1$ or (3) is infeasible.

Proof. Suppose $\varphi_o^* = 1$. Since DMU_j $(j \in E_o)$ exhibit CRS, by RC2, DMU_o has an optimal solution to (1) with $\sum_{j \in E_o} \lambda_j^* = 1$ and $\theta^* = 1$. Therefore DMU_o exhibits CRS.

Next, if $DMU_o = (x_o, y_o)$ exhibits CRS, then $DMU_o\ (\delta) = (\delta x_o, y_o)$ also exhibits CRS under (1), where $\theta^* \le \delta < +\infty$, and θ^* is the optimal value to (1) when evaluating DMU_o. Suppose $\varphi_o^* \neq 1$. Let $\lambda_j^* = \varphi_o^* \lambda_j$, where λ_j^* $(j \in E_o)$ is an optimal solution to (3) associated with φ_o^*. We have

$$\sum_{j \in E_o} \lambda_j x_{ij} \le \frac{1}{\varphi_o^*} x_{io} \quad i = 1, 2, \ldots, m;$$
$$\sum_{j \in E_o} \lambda_j y_{rj} \ge y_{ro} \qquad r = 1, 2, \ldots, s;$$
$$\sum_{j \in E_o} \lambda_j = \frac{1}{\varphi_o^*} < 1 \qquad j \in E_o.$$

If $\frac{1}{\varphi_o^*} \le \theta^*$, then the optimality of θ^* is violated. If $\frac{1}{\varphi_o^*} > \theta^*$, then let $\frac{1}{\varphi_o^*} = \delta\theta^*$. Obviously, $\frac{1}{\varphi_o^*}$ is the optimal value to (1) when evaluating $(\delta x_o, y_o)$, where $\theta^* \le \delta < +\infty$. However, $\sum_{j \in E_o} \lambda_j < 1$ violating RC1. Therefore $\varphi_o^* = 1$ must hold. This completes the proof of (a).

If $\varphi_o^* < 1$, then the optimal value to (3) is equal to one for $DMU_o' = (x_o, \varphi_o^* y_o)$. From (a), we know that CRS prevail for DMU_o'. Thus DMU_o can not exhibit IRS. (We can not decrease the outputs and cause a IRS DMU to exhibit CRS). Therefore DRS prevail for DMU_o. This completes the *if* part of (b).

From lemma 3 and (a), we know that if (3) is infeasible, then IRS must prevail for DMU_o. If $\varphi_o^* > 1$, then similar to the proof of *if* part of (b), DMU_o can not exhibit DRS. Therefore IRS prevail for DMU_o. This completes the proof of *if* part of (c).

The *only if* part of (b) and (c) follows directly from the mutually exclusive and exhaustive conditions specified in the theorem. ■

Remark 2. Under RC1, any proportion of output change in a CCR-inefficient DMU exhibiting CRS will alter its RTS nature. The *only if* parts of (b) and (c) are true without RC1. We see that if $\varphi_o^* < 1$, then DMU_o will also be termed as having DRS by (2). Thus (3) finds out the identical DRS regions under (1) and (2). Finally note that this theorem gives an alternative approach for estimating the RTS.

THEOREM 2 *Suppose DMU_o exhibits DRS. If $\varphi_o^* < \beta \leq 1$ then the DRS classification still holds for a proportional decrease of amount β.*

Proof. Suppose the outputs of DMU_o decrease to $\hat{\beta} y_{ro}$ $(r = 1, 2, \ldots, s)$ where $\varphi_o^* < \hat{\beta} \leq 1$. If the RTS estimate is no longer held, then the RTS on DMU_o will be CRS or IRS.

Consider the following linear programming problem:

$$\begin{aligned}
& \hat{\varphi}_o^* = \max \hat{\varphi}_o \\
s.t.\ & \sum_{j \in E_o} \lambda_j x_{ij} \leq x_{io} \qquad i = 1, 2, \ldots, m; \\
& \sum_{j \in E_o} \lambda_j y_{rj} \geq \hat{\varphi}_o \hat{\beta} y_{ro} \quad r = 1, 2, \ldots, s; \\
& \sum_{j \in E_o} \lambda_j = 1 \\
& \lambda_j \geq 0 \qquad\qquad j \in E_o
\end{aligned} \tag{5}$$

Obviously, (5) has a feasible solution of λ_j $(j \in E_o)$ and $\hat{\varphi}_o = \frac{\varphi_o^*}{\hat{\beta}}$. Thus either $\hat{\varphi}_o^* = 1$ or $\hat{\varphi}_o^* > 1$ will violate the optimality of φ_o^*. Therefore DRS still prevail on DMU_o. ■

THEOREM 3 *Suppose DMU_o exhibits IRS and (3) is feasible. If $1 \leq \alpha < \varphi_o^*$ then the IRS classification continues to hold for an increase of amount α.*

Proof. The proof is analogous with that of theorem 2 and is omitted. ■

Thus when (3) is feasible, the optimal value to (3) determines the maximum possible output proportional change factors for IRS and DRS DMUs which preserve their RTS classification.

If (3) is infeasible, then these IRS DMUs do not belong to T_o. In this situation, we consider the output-based CCR model (2) to determine the maximum perturbation.

THEOREM 4 *Suppose (3) is infeasible. Let α satisfy $1 \leq \alpha < \vartheta^*$, where ϑ^* is the optimal value to (2) when evaluating DMU_o. Then IRS continue to hold for DMU_o for an increase of amount α.*

Proof. Suppose the output of DMU_o is increased to $\hat{\alpha}$, where $1 \le \hat{\alpha} < \vartheta^*$, and the resulting DMU exhibits CRS or DRS. Then we have an optimal solution, λ_j^* $(j \in E_o)$ and θ^* to (1) such that

$$\sum_{j \in E_o} \lambda_j^* x_{ij} \le \theta^* x_{io} \quad i = 1, 2, \ldots, m;$$

$$\sum_{j \in E_o} \lambda_j^* y_{rj} \le \hat{\alpha} y_{ro} \quad r = 1, 2, \ldots, s;$$

$$\sum_{j \in E_o} \lambda_j^* \ge 1 \quad j \in E_o.$$

Obviously, $\lambda_j = \frac{\lambda_j^*}{\sum_{j \in E_o} \lambda_j^*}$ and $\varphi_o = \frac{\hat{\alpha}}{\sum_{j \in E_o} \lambda_j^*}$ is a feasible solution to (3) violating the infeasibility of (3). ■

Remark 3. In this situation, DMU_o is moved toward the CCR frontier. Theorem 4 indicates that if (3) is infeasible then the input-based and output-based DEA techniques both classify DMU_o as IRS. Thus (3) is also an indicator of the identical IRS regions yielded by (1) and (2).

It can be seen that measure (3) not only analyzes the stability of the RTS classifications but also gives the RTS classifications. i.e., both the RTS classification of a specific DMU and its stability can be obtained from one model.

The above discussion only considers proportional output changes. In fact, we can easily consider non-proportional changes. Note that if DMU_o exhibits DRS, then $\varphi_o^* < 1$ in (3). This implies that this DMU_o is BCC-extreme-efficient and in set E (the DMU group $J_o \subseteq J$ now consists of DMU_j $(j \in E_o)$ and DMU_o). Therefore we may directly employ the technique in Zhu (1996) and Seiford and Zhu (1998a, 1998b) to determine the possible output decreases $\beta_r \le 1$ defined in $\hat{y}_{ro} = \beta_r y_{ro}$ $(r = 1, 2, \ldots, s)$ which preserve the DRS classification of DMU_o (see Charnes and Neralic (1990) for an alternate approach to sensitivity analysis in DEA).

$$\begin{aligned} & \beta_k^* = \max \beta_k \\ s.t. & \sum_{j \in E_o} \lambda_j x_{ij} \le x_{io} \qquad i = 1, 2, \ldots, m; \\ & \sum_{j \in E_o} \lambda_j y_{kj} \ge \beta_k y_{ko} \\ & \sum_{j \in E_o} \lambda_j y_{rj} \ge y_{ro} \qquad r \ne k \\ & \sum_{j \in E_o} \lambda_j = 1 \\ & \lambda_j \ge 0 \qquad j \in E_o \end{aligned} \tag{6}$$

Obviously $\beta_k^* \le 1$ $(k = r = 1, 2, \ldots s)$. Model (6) gives the possible maximum decrease rate for each single output which allows DRS to prevail for DMU_o.

THEOREM 5 *DRS continue to hold for DMU_o with individual decreases β_r, if and only if $(\beta_1, \ldots, \beta_s) \in \Lambda$, where $\Lambda = \{(\beta_1, \ldots, \beta_s) \mid \beta_r^* < \beta_r \le 1, r = 1, \ldots, s$ and $A_1\beta_1 +$*

$\cdots + A_s\beta_s > 1\}$ *and the parameters* A_r *can be determined by the following system of equations:*

$$\begin{cases} \beta_1^* A_1 + \quad A_2 + \cdots + \quad A_s = 1 \\ \quad A_1 + \beta_2^* A_2 + \cdots + \quad A_s = 1 \\ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \quad A_1 + \quad A_2 + \cdots + \beta_s^* A_s = 1 \end{cases} \tag{7}$$

Proof. From theorem 1, we know that the DRS nature of DMU_o stays unchanged *if and only if* DMU_o is still BCC-extreme-efficient with reference to J_o . Therefore the proof is the same as that in Zhu (1996). ■

Now if DMU_o exhibits IRS, then the above development of individual changes again applies. If (3) is feasible, then we use (6) to calculate the maximum increase rate, $\alpha_r \geq 1$ defined in $\hat{y}_{ro} = \alpha_r y_{ro}$, for each single output to allow DMU_o to exhibit IRS; if (3) is infeasible, we use a (2)-like formulation, i.e., we delete the convexity constraint in (6). That is,

$$\begin{aligned} & \alpha_k^* = \max \alpha_k \\ s.t.\ & \sum_{j \in E_o} \lambda_j x_{ij} \leq x_{io} && i = 1, 2, \ldots, m; \\ & \sum_{j \in E_o} \lambda_j y_{kj} \geq \alpha_k y_{ko} \\ & \sum_{j \in E_o} \lambda_j y_{rj} \geq y_{ro} && r \neq k \\ & \pi_1 \sum_{j \in E_o} \lambda_j = \pi_2 \\ & \lambda_j \geq 0 && j \in E_o \end{aligned} \tag{8}$$

If (3) is feasible, let $\pi_1 = \pi_2 = 1$, i.e., (8) is identical to (6); if (3) is infeasible, let $\pi_1 = \pi_2 = 0$, i.e., (8) is developed from (2). Obviously, $\alpha_k^* \geq 1$ ($k = r = 1, 2, \ldots, s$).

THEOREM 6 *IRS continue to hold for* DMU_o *if and only if* $(\alpha_1, \ldots, \alpha_s) \in \tilde{\Lambda}$, *where* $\tilde{\Lambda} = \{(\alpha_1, \ldots, \alpha_s) \mid 1 \leq \alpha_r < \alpha_r^*, r = 1, \ldots, s$ *and* $A_1\alpha + \cdots + A_s\alpha_s < 1\}$ *and the parameters of* A_r *can be determined by using* α_r^* *instead of* β_r^* *(*$r = 1, 2, \ldots, s$*) in (7).*

Proof. The proof is similar to that of theorem 5. But in this case, DMU_o is moved toward the boundary of T_o from the inside of T_o when (3) is feasible, or it is moved toward the CCR frontier when (3) is infeasible. ■

Theorems 5 and 6 provide the necessary and sufficient conditions for preserving the RTS classification of DMU_o.

(3.B) Sensitivity of the RTS Classifications in Terms of (2)

We now consider input perturbations instead of output changes in DMU_o. Note that under (2), if DMU_o exhibits DRS, then increases in inputs can not change its DRS nature. Likewise, if DMU_o exhibits IRS, decreases in inputs can not change its IRS nature unless DMU_o reaches the CCR frontier. Therefore we only consider input increases and decreases, respectively, for IRS and DRS DMUs.

Suppose that DMU_o may proportionally increase and decrease its inputs, respectively, by $\eta \geq 1$ and $\xi \leq 1$, up to ηx_{io} and ξx_{io} $(i = 1, 2, \ldots, m)$ while its RTS classification still holds.

In order to calculate η and ξ, we define the following measure:

$$\phi_o^* = \min\{\phi_o\colon\ (\phi_o x_o, y_o) \in T_o\} \tag{9}$$

where (x_o, y_o) represents the input and output vector for DMU_o and ϕ_o^* can be calculated as the solution to the linear programming problem:

$$\begin{aligned} &\phi_o^* = \min \phi_o \\ s.t.\ &\sum_{j\in E_o} \lambda_j x_{ij} \leq \phi_o x_{io} \quad i = 1, 2, \ldots, m; \\ &\sum_{j\in E_o} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1, 2, \ldots, s; \\ &\sum_{j\in E_o} \lambda_j = 1 \\ &\quad \lambda_j \geq 0 \qquad\qquad j \in E_o \end{aligned}$$

Note that the above model is the input-based BCC model if $E_o = J$. As for theorem 1, four possible cases are associated with (9), that is, $\phi_o^* = 1$, $\phi_o^* > 1$, $\phi_o^* < 1$ or (9) is infeasible, and one can obtain the following RTS characterization.

THEOREM 7

(a) CRS prevail for DMU_o if and only if $\phi_o^* = 1$;

(b) IRS prevail for DMU_o if and only if $\phi_o^* > 1$;

(c) DRS prevail for DMU_o if and only if $\phi_o^* < 1$ or (9) is infeasible.

Remark 4. Obviously, no input changes are allowed in DMU_o if CRS prevail when RC1 holds. The *only if* parts of (b) and (c) are true without RC1. If (9) is infeasible, then DRS must prevail on DMU_o. If $\phi_o^* > 1$, then DMU_o will also be termed as having IRS by (1).

I.e., (9) finds out the identical IRS regions generated by (1) and (2). This theorem also gives an alternative RTS method under the output-based DEA technique.

Furthermore we have:

THEOREM 8 *Suppose DMU_o exhibits IRS. For an input increase of amount η, where $1 \leq \eta < \phi_o^*$ then the IRS classification continues to hold.*

THEOREM 9 *Suppose DMU_o exhibits DRS and (9) is feasible. For an input decrease amount of ξ, where $\phi^* < \xi \leq 1$, the DRS classification continues to hold.*

Thus when (9) is feasible, the optimal value to (9) determines the maximum possible input proportional change factors for IRS and DRS DMUs which preserve their RTS classifications.

If (9) is infeasible, then these DRS DMUs do not belong to T_o. In this situation, we consider the input-based CCR model (1).

THEOREM 10 *Suppose (9) is infeasible. For an input decrease amount of ξ, where $\theta^* < \xi \leq 1$ then DRS still prevail for DMU_o, where θ^* is the optimal value to (1) when evaluating DMU_o.*

Remark 5. This theorem indicates that if (9) is infeasible then the input-based and output-based DEA techniques both declare DMU_o as DRS. Thus (9) also indicates the identical DRS regions yielded by (1) and (2).

From the above discussion, we see that measure (9) can also be used to estimate the RTS classification for DMU_o in addition to its role in sensitivity analysis.

We may also consider non-proportional changes in all of the inputs of DMU_o . Note that if DMU_o exhibits IRS, then $\phi_o^* > 1$ in (9) indicating that DMU_o is BCC-extreme-efficient with reference to J_o. Therefore, we can employ the method in Zhu (1996) and Seiford and Zhu (1998a, 1998b) to determine the stability region of IRS classification when all the inputs increase non-proportionally. On the other hand, if DMU_o exhibits DRS, then either $\phi_o^* < 1$ or $\theta^* < 1$. Therefore we can determine the possible input decrease region before DMU_o is moved onto the boundary of T_o. In fact, these developments are analogous with those described in the output change case of (3.A), and we leave the details to the interested readers.

4. General Situation

The previous developments assume that (1) and (2) have $\sum_{j \in E_o} \lambda_j^* = 1$ in all possible optimal solutions for CRS DMUs. If this does not hold, then $\sum_{j \in E_o} \lambda_j^*$ may also be either greater or less than one for the CRS DMUs associated with different optimal basis sets. Consequently, φ_o^* in (3) (or ϕ_o^* in (9)) may also be larger or smaller than one in the different optimal basis sets associated with E_o. Therefore, some data perturbations in the CRS DMUs can be allowed. In this section, we will further discuss the RTS sensitivity analysis without requiring RC1 and RC2. Note that if $\varphi_o^* > 1$ for DRS DMUs (or $\phi_o^* < 1$ for IRS DMUs), then RC2 is violated.

(4.A) Stability of the RTS Classifications in Terms of (1)

Suppose DMU_o exhibits CRS. On the basis of E_o, we define the following two linear programming problems:

$$
\begin{aligned}
(\tau_o^*)^{-1} = \min \sum_{j \in E_o} \hat{\lambda}_j& \\
s.t. \sum_{j \in E_o} \hat{\lambda}_j x_{ij} \le \theta^* x_{io} & \qquad i = 1, 2, \ldots, m; \\
\sum_{j \in E_o} \hat{\lambda}_j y_{rj} \ge y_{ro} & \qquad r = 1, 2, \ldots, s; \\
\hat{\lambda}_j \ge 0 & \qquad j \in E_o
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
(\sigma_o^*)^{-1} = \max \sum_{j \in E_o} \hat{\lambda}_j& \\
s.t. \sum_{j \in E_o} \hat{\lambda}_j x_{ij} \le \theta^* x_{io} & \qquad i = 1, 2, \ldots, m; \\
\sum_{j \in E_o} \hat{\lambda}_j y_{rj} \ge y_{ro} & \qquad r = 1, 2, \ldots, s; \\
\hat{\lambda}_j \ge 0 & \qquad j \in E_o
\end{aligned} \tag{11}
$$

where θ^* is the optimal value to (1) when evaluating DMU_o .

Since DMU_o exhibits CRS, therefore $\tau_o^* = (\sum_{j \in E_o} \hat{\lambda}_j^*)^{-1} \ge 1$ ($\sigma_o^* = (\sum_{j \in E_o} \hat{\lambda}_j^*)^{-1} \le 1$) where $\hat{\lambda}_j^*$ ($J \in E_o$) represent optimal solutions to (10) ((11)). Obviously $\hat{\lambda}_j^*$ ($j \in E_o$) with $\sum_{j \in E_o} \hat{\lambda}_j^* \le 1$ ($\sum_{j \in E_o} \hat{\lambda}_j^* \ge 1$) is also an optimal solution to (1) ((2)).

THEOREM 11 *Suppose DMU_o exhibits CRS. If $\chi \in \mathbf{R}^{\mathrm{CRS}} = \{\chi\colon \min\{1, \sigma_o^*\} \le \chi \le \max\{1, \tau_o^*\}\}$. Then the CRS classification continues to hold, where χ represents a proportional change of all outputs, $\hat{y}_{ro} = \chi y_{ro}$ ($r = 1, 2, \ldots, s$) and, τ_o^* and σ_o^* are defined in (10) and (11) respectively.*

Proof. By Thrall and Banker (1992), we know that $(\theta^* \tau_o^* x_o, \tau_o^* y_o)$ and $(\theta^* \sigma_o^* x_o, \sigma_o^* y_o)$ both exhibit CRS. Consequently, $(x_o, \tau_o^* y_o)$ and $(x_o, \sigma_o^* y_o)$ exhibit CRS. Therefore if $\min\{1, \sigma_o^*\} \le \chi \le \max\{1, \tau_o^*\}$, then DMU_o $(= x_o, \chi y_o)$ still exhibits CRS. ■

Remark 6. If $\sum_{j \in E_o} \lambda_j \ge 1$ for all alternate optima to (1), then $\sigma_o^* = \varphi^* = 1$ and no proportional output increase is allowed. If $\sum_{j \in E_o} \lambda_j \le 1$ for all alternate optima to (1), then $\tau_o^* = 1$ and no proportional output decrease is allowed. If $\sum_{j \in E_o} \lambda_j$ can be equal to, larger than, or less than one, then both proportional increases and decreases of output are possible. (See the example in Appendix B.) If RC2 holds, then $\mathbf{R}^{\mathrm{CRS}} = \{\chi\colon \min\{1, \sigma_o^*\} \le \chi \le$

$\max\{1, \tau_o^*, \varphi_o^*\}$. Furthermore, if only RC1 is violated, theorem 1 (a) should be modified to read: CRS prevail for DMU_o *if and only if* there exist some E_o such that $\varphi^* = 1$ in (3).

Next we discuss the RTS sensitivity analysis for IRS DMUs. If DMU_o exhibits IRS, then $\sum_{j \in E_o} \lambda_j < 1$ in all optimal solutions to (1). Thus $\sigma_o^* > 1$ in (11).

THEOREM 12 *Suppose DMU_o exhibits IRS. The IRS classification continues to hold for $\alpha \in \mathbf{R}^{IRS} = \{\alpha\colon 1 \le \alpha < \sigma_o^*\}$, where α represents the proportional increase of all outputs, $\hat{y}_{ro} = \alpha y_{ro}$ $(r = 1, 2, \ldots, s)$ and σ_o^* is defined in (11)*

Proof. Suppose $DMU_o' = (x_o, \alpha\, y_o)$ and DMU_o' exhibits CRS or DRS. Then $DMU_o'' = (\alpha\,\theta^* x_o, \alpha\, y_o)$, must also exhibit CRS or DRS. Furthermore, we have

$$\begin{aligned}
&\sum_{j \in E_o} \lambda_j^* x_{ij} \le \gamma^* \alpha \theta^* x_{io} \le \alpha \theta^* x_{io} && I = 1, 2, \ldots, m;\\
&\sum_{j \in E_o} \lambda_j^* y_{rj} \le \alpha y_{ro} && r = 1, 2, \ldots, s;\\
&\sum_{j \in E_o} \lambda_j^* \ge 1 && j \in E_o.
\end{aligned}$$

where γ^* is the optimal value to (1) when evaluating DMU_o''. Obviously, $\frac{\lambda_j^*}{\alpha}$ $(j \in E_o)$ is a feasible solution to (11). Thus $\frac{\sum_{j \in E_o} \lambda_j^*}{\alpha} \ge \frac{1}{\alpha} > \frac{1}{\sigma_o^*}$ violating the optimality of (11). ■

From the proof of theorem 4, we know that theorem 4 holds in the absence of RC1 and RC2. Therefore if (3) is infeasible for DMU_o, then the RTS stability region is $\mathbf{R}^{\mathrm{IRS}} = \{\alpha\colon 1 \le \alpha < \max\{\vartheta^*, \sigma_o^*\}\}$, where ϑ^* is the optimal value to (2).

Finally, we consider the DRS DMUs.

LEMMA 4 *If DMU_o exhibits DRS in (1), then DMU_o must exhibit DRS in (2).*

Proof. Suppose DMU_o exhibits CRS or IRS in (2). Then by lemma 2, we have $\sum_{j \in E_o} \lambda_j^* \le \theta^* \le 1$, where λ_j^* $(j \in E_o)$ and θ^* is an optimal solution to (1). Since DMU_o exhibits DRS in (1), therefore $\sum_{j \in E_o} \lambda_j^* > 1$ in all alternative optimal solutions to (1). Thus $\theta^* > 1$, a contradiction. ■

The following lemma is obvious. Note that φ_o does not necessarily represent the optimal value to (3).

LEMMA 5 *If CRS prevail for DMU_o, then there exists some E_o such that $\varphi_o = 1$ in (3).*

THEOREM 13 *Suppose DMU_o exhibits DRS and $\varphi_o^* < 1$. Then the DRS classification continues to hold for $\varphi_o^* < \beta \le 1$, where β represents the proportional change of all outputs, $\hat{y}_{ro} = \beta y_{ro}$ $(r = 1, 2, \ldots, s)$ and φ_o^* is the optimal value to (3).*

Proof. By lemma 4, DMU_o exhibits DRS under (2). Next, let $DMU'_o = (x_o, \beta y_o)$. Then DMU'_o still exhibits DRS under (2). Thus

$$\begin{aligned} &\sum_{j\in E_o} \lambda_j^* \le x_{io} && i = 1, 2, \ldots, m;\\ &\sum_{j\in E_o} \lambda_j^* y_{rj} \le \vartheta^* \beta y_{ro} && r = 1, 2, \ldots, s;\\ &\sum_{j\in E_o} \lambda_j^* > 1 && j \in E_o. \end{aligned}$$

where ϑ^* is the optimal value to (2) when evaluating DMU'_o.

If DMU'_o exhibits IRS in (1), then, by lemma 2, $\frac{\sum_{j\in E_o}\lambda_j^*}{\vartheta^*} < 1$. Thus $\varphi_o = \frac{\vartheta^*\beta}{\sum_{j\in E_o}\lambda_j^*} > \varphi_o^*$ is a feasible solution to (3) which violates the optimality of φ_o^*.

If DMU'_o exhibits CRS, then, by lemma 5, we have $\varphi_o = 1$ when calculating (3) for DMU'_o. Thus $\beta > \varphi_o^*$ which violates the optimality of φ_o^*. ■

However, one may also use the optimal value to (10), $\tau_o^* < 1$, to determine the stability region, particularly in the case of $\varphi_o^* > 1$ for a DRS DMU_o. This is characterized by the following theorem.

THEOREM 14 *Suppose DMU_o exhibits DRS. Then the DRS classification continues to hold for $\beta \in \mathbf{R}^{\mathrm{DRS}} = \{\beta\colon \tau_o^* < \beta \le 1\}$, where β represents the proportional change of all outputs, $\hat{y}_{ro} = \beta y_{ro}$ $(r = 1, 2, \ldots, m)$ and τ_o^* is defined in (10).* [3]

Proof. The proof is analogous with that of theorem 12 and is omitted. ■

Finally, we can use τ_o^* and σ_o^* to estimate the RTS classifications.

THEOREM 15

(a) *CRS prevail for DMU_o if and only if $\sigma_o^* \le 1 \le \tau_o^*$;*

(b) *DRS prevail for DMU_o if and only if $\tau_o^* < 1$;*

(c) *IRS prevail for DMU_o if and only if $\sigma_o^* > 1$.*

Proof. The *only if* parts of (b) and (c) are obvious. Next, if $\tau_o^* < 1$, then $\sum_{j\in E_o} \hat{\lambda}_j^* > 1$, where $\sum_{j\in E_o} \hat{\lambda}_j^*$ is the optimal value to (10). This indicates that $\sum_{j\in E_o} \lambda_j^* > 1$ in all alternative optimal solutions to (1). Thus DRS prevail for DMU_o. This completes the proof of the *if* part of (b). The proof of the *if* part of (c) is similar. The *if* and the *only if* parts of (a) follow directly. ■

(4.B) Stability of the RTS Classifications in Terms of (2)

Consider the following two linear programming models:

$$
\begin{aligned}
&(\tilde{\tau}_o^*)^{-1} = \min \sum_{j\in E_o} \tilde{\lambda}_j \\
s.t.\ &\sum_{j\in E_o} \tilde{\lambda}_j x_{ij} \le x_{io} \qquad i = 1, 2, \ldots, m; \\
&\sum_{j\in E_o} \tilde{\lambda}_j y_{rj} \ge \vartheta^* y_{ro} \qquad r = 1, 2, \ldots, s; \\
&\tilde{\lambda}_j \ge 0 \qquad j \in E_o.
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&(\tilde{\sigma}_o^*)^{-1} = \max \sum_{j\in E_o} \tilde{\lambda}_j \\
s.t.\ &\sum_{j\in E_o} \tilde{\lambda}_j x_{ij} \le x_{io} \qquad i = 1, 2, \ldots, m; \\
&\sum_{j\in E_o} \tilde{\lambda}_j y_{rj} \ge \vartheta^* y_{ro} \qquad r = 1, 2, \ldots, s; \\
&\tilde{\lambda}_j \ge 0 \qquad j \in E_o.
\end{aligned}
\tag{13}
$$

where ϑ^* is the optimal value to (2) when evaluating DMU_o.

Suppose DMU_o exhibits CRS. Then $\sum_{j\in E_o} \tilde{\lambda}_j^* \le 1$ in (12) and $\sum_{j\in E_o} \tilde{\lambda}_j^* \ge 1$ in (13), i.e., $\tilde{\tau}_o^* \ge 1$ and $\tilde{\sigma}_o^* \le 1$ respectively. Similar to theorem 11, we obtain the following.

THEOREM 16 *Suppose DMU_o exhibits CRS. If $\gamma \in \mathbf{R}^{\text{CRS}} = \{\gamma\colon \min\{1, \sigma_o^*\} \le \chi \le \max\{1, \tau_o^*\}\}$. Then the CRS classification continues to hold, where γ represents the proportional change of all inputs, $\hat{x}_{io} = \gamma x_{io}$ ($i = 1, 2, \ldots, m$) and $\tilde{\tau}_o^*$ and $\tilde{\sigma}_o^*$ are defined in (12) and (13) respectively.*

Remarks. If $\sum_{j\in E_o} \lambda_j \ge 1$ in all alternate optima to (2), then $\tilde{\sigma}^* = 1$ and no proportional input increase is allowed. If $\sum_{j\in E_o} \lambda_j \le 1$ in all alternate optima to (2), then $\tilde{\tau}_o^* = 1$ and $\phi^* = 1$ and no proportional input decrease is allowed. If $\sum_{j\in E_o} \lambda_j$ can be equal to, larger than, or less than one, then both proportional input increase and decrease are possible. In this situation, E_o in (9) is identified by the different optimal basis sets associated with nonzero lambdas in (2). If RC2 holds, then $\mathbf{R}^{\text{CRS}} = \{\gamma\colon \min\{1, \sigma_o^*, \phi_o^*\} \le \chi \le \max\{1, \tau_o^*\}\}$. Furthermore, if only RC1 is violated, theorem 8 (a) should be modified to read: CRS prevail for DMU_o *if and only if* there exists a E_o such that $\phi^* = 1$ in (9).

If DMU_o exhibits DRS, then $\sum_{j\in e_o} \tilde{\lambda}_j^* > 1$, i.e., $\tilde{\tau}^* < 1$ in (12) and similar to theorem 12, we obtain

THEOREM 17 *Suppose DMU_o exhibits DRS. The DRS classification continues to hold for $\xi \in \mathbf{R}^{\text{DRS}} = \{\xi\colon \tilde{\tau}_o^* < \xi \le 1\}$, where ξ represents the proportional decrease of all inputs, $\hat{x}_{io} = \xi x_{io}$ ($i = 1, 2, \ldots, m$) and $\tilde{\tau}_o^*$ is defined in (12).*

Theorem 10 holds for the situation without RC1 and RC2. Therefore if (9) is infeasible, then the RTS stability region is $\mathbf{R}^{\text{DRS}} = \{\xi\colon \min\{\theta^*, \tilde{\tau}_o^*\} < \xi \le 1\}$, where θ^* is the optimal value to (1) when evaluating DMU_o.

For IRS DMUs, we have

THEOREM 18 *Suppose DMU_o exhibits IRS and $\phi_o^* > 1$. Then the IRS classification continues to hold for $1 \leq \eta < \phi_o^*$, where η represents the proportional change of all inputs, $\hat{x}_{io} = \eta x_{io}$ $(i = 1, 2, \ldots, m)$ and ϕ_o^* is the optimal value to (9).*

However, one may also use the optimal value to (10), $\tau_o^* > 1$, to determine the IRS stability region, particularly in the case of $\varphi_o^* > 1$ for DMU_o. This is characterized by the following theorem.

THEOREM 19 *Suppose DMU_o exhibits IRS. Then the IRS classification continues to hold for $\eta \in \mathbf{R}^{\mathrm{IRS}} = \{\eta\colon 1 \leq \eta < \tilde{\sigma}_o^*\}$, where η represents the proportional change of all inputs, $\hat{x}_{io} = \eta x_{io}$ $(i = 1, 2, \ldots, m)$ and $\tilde{\sigma}_o^*$ is defined in (13).*

5. Concluding Remarks

The estimation of RTS in DEA provides important information on scale efficiency and on improving the performance of DMUs. One would like to determine the movement of CCR-inefficient DMUs onto the frontier in improving directions. Therefore, the sensitivity of the RTS classifications is extremely important for empirical applications.

The current paper develops linear programming techniques for studying the sensitivity of RTS estimation obtained from the input-based and output-based DEA methods respectively. The only information needed is the optimal basis set (or facet DMUs) obtained from evaluating a specific DMU_o by the original CCR model. The sensitivity analysis method can easily be applied to real world data sets via existing DEA codes. The sensitivity analysis approach is developed for handling situations when output perturbations occur in DMU_o under the input-based DEA model and input perturbations under the output-based DEA model.

In addition, our sensitivity analysis approach also gives an alternative method to classify RTS for each DMU, and it can be employed to identify the identical RTS regions obtained from input-based and output-based DEA models respectively. In particular, if the CCR model yields the unique optimal lambda solution, the summation of the lambda variables can be directly used to define the lower or upper boundary for the RTS stability regions.

In the current study, we only consider data perturbations for CCR inefficient DMUs. Note that the movement of CCR efficient DMUs may also change the RTS classification. By the ray unboundedness assumption of the CCR model, the effect on the RTS classification by the movement of CCR efficient DMUs along the frontier is straightforward. However, one possible future research subject would be to examine situations where both CCR efficient DMUs and CCR inefficient DMUs were perturbed.

Appendix A

We provide a simple numerical example which illustrates the sensitivity analysis of RTS results obtained from the input-based RTS method. An artificial set of data, containing 6

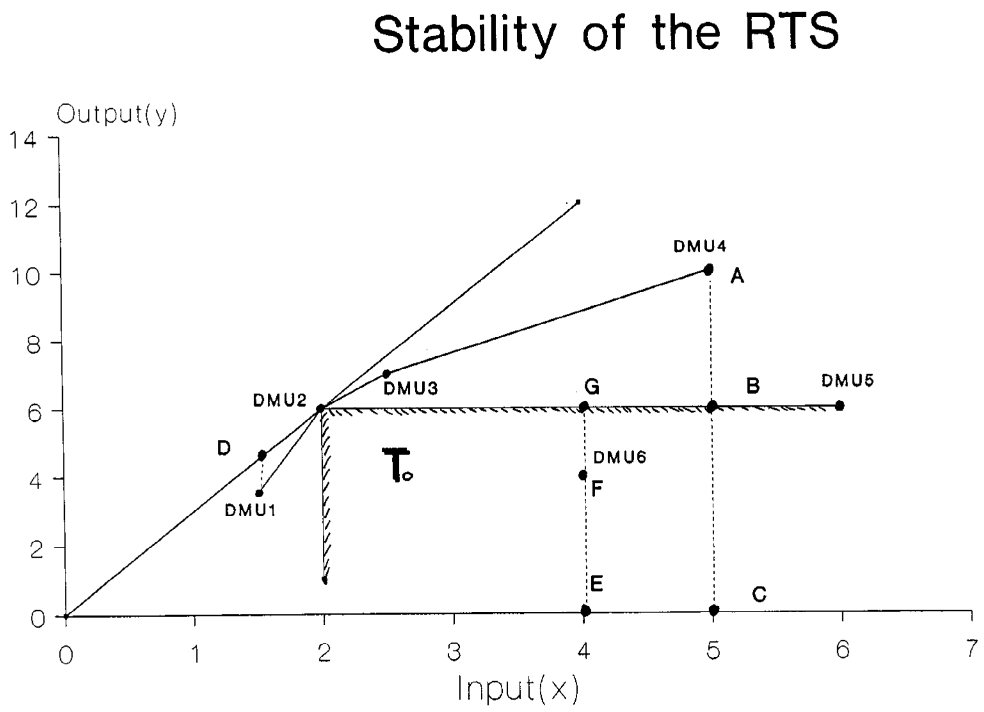

Figure A. Stability of the RTS.

Table A-1. Sample data set.

DMU	1	2	3	4	5	6
input (x_j)	1.5	2	2.5	5	6	4
output (y_j)	3.5	6	7	10	6	4

DMUs with a single output and a single input, was generated. Raw data are provided in Table A.1. Figure A displays the set T_o used in measure (3).

From Figure A, we know that DMU2 is CCR-efficient. DMU5 exhibits CRS, DMUs 1 and 6 exhibit IRS, and DMUs 3 and 4 exhibit DRS. Table 2 provides the sensitivity results of the RTS estimation on DMUs 1, 3, 4, 5 and 6 when the output is changed.

Table A-2. RTS sensitivity results.

DMU	1	3	4	5	6
Sensitivity	$\vartheta^* = 9/7^{\#}$	$\varphi_o^* = 6/7$	$\varphi_o^* = 3/5$	$\varphi_o^* = 1$	$\varphi_o^* 3/2$
Result	$\alpha \in [1, 9/7)$	$\beta \in (6/7, 1]$	$\beta \in (3/5, 1]$	$\alpha = \beta = 1$	$\alpha \in [1, 3/2)$

(3) (or (A.1)) is infeasible for DMU1.

For this data set, $E_o = \{\text{DMU2}\}$. Thus (3) can be written as

$$\begin{aligned} &\varphi_o^* = \min \varphi_o \\ s.t.\ &2\lambda \leq x_o \\ &6\lambda \geq \varphi_o y_o \\ &\lambda = 1. \end{aligned} \tag{A.1}$$

where (x_o, y_o) represents one of DMUs 1, 3, 4, 5 and 6.

DMUs 3 and 4 exhibit DRS, so we have $\varphi_o^* < 1$. The value of φ_o^* gives the maximum distance from DMU3 or DMU4 to the boundary (GB) of T_o. For instance, in DMU4, $\varphi_o^* = \frac{AB}{AC} = \frac{y_2}{y_4} < 1$. The two DMUs can not be moved into T_o or onto the boundary of T_o and still maintain the DRS classification. For example, if the proportional output decrease is in the interval of $(3/5, 1]$, then DMU4 still exhibits DRS, otherwise CRS and then finally IRS will prevail.

Although both DMUs 1 and 6 exhibit IRS, (A.1) is infeasible for DMU1. Thus, we calculate the output-based CCR model (2) and obtain $\vartheta^* = 9/7$ where point D is the referent DMU. This means that the output of DMU1 must be greater than 9/2 before CRS hold for DMU1. In fact DMU1 will move onto the CCR frontier (the ray OD in Figure A). Note that DMU1 $\notin T_o$ and both (1) and (2) give the same RTS estimation of IRS. Since DMU6 belongs to T_o, we have the optimal value of $\varphi_o^* = \frac{EF}{FG} = \frac{y_2}{y_6} = \frac{3}{2}$ which means that if the output increase factor is less than 3/2, then the current RTS classification (IRS) will continue to hold. Geometrically DMU6 can not be moved outside of T_o under output increases by more than $3/2$.

In addition, note that $\varphi_o^* = 1$ for DMU5 where CRS prevail. This means that any change in output will cause DMU5 to leave the boundary of T_o.

Appendix B

Consider an example taken from Zhu and Shen (1995) with m = 2, s = 1, n = 4 and

$$\begin{pmatrix} x_{1j} \\ x_{2j} \\ y_j \end{pmatrix} = \begin{bmatrix} 0.1 & 2 & 40 & 3 \\ 0.25 & 2 & 10 & 2 \\ 0.1 & 1 & 10 & 1 \end{bmatrix}$$

DMUs 1, 2 and 3 are CCR-efficient and are on the same efficient facet given by $x_1 + 2x_2 = 6y$. Obviously, RC2 is satisfied. DMU4 is inefficient with $\theta^* = 6/7$. We obtain

Table C-1. RTS Stability regions for the Chinese cities.

Cities	RTS	E_o	φ_o^*	τ_o^*, σ_o^*	Stability Region
2	DRS	6, 8, 21	0.46955	0.36258	(0.36258, 1]
3	DRS	1, 21, 24	1.31533	0.50761	(0.50761, 1]
4	DRS	8, 21, 24, 26	0.85771	0.70771	(0.70771, 1]
5	DRS	8, 24, 26	0.79834	0.62305	(0.62305, 1]
7	IRS	6, 8, 21, 26	1.48776	1.20627	[1, 1.20627)
9	IRS	1, 8, 21	1.37427	1.20627	[1, 1.20627)
10	IRS	6, 8, 21	1.59622	1.51057	[1, 1.51057)
11	IRS	6, 8, 21, 26	1.24228	1.06838	[1, 1.06838)
12	IRS	6, 8, 26	1.21293	1.03842	[1, 1.03842)
13	IRS	1, 8, 21	1.97062	2.88184	[1, 2.88184)
14	IRS	1, 8, 21	1.30574	1.16279	[1, 1.16279)
15	IRS	8, 21, 24, 26	1.12570	1.06045	[1, 1.06045)
16	IRS	1, 8, 21	infeasible 1.67410	2.95858	[1, 2.95858)
17	IRS	6, 8, 21, 26	1.55605	1.76367	[1, 1.76367)
18	IRS	8, 21, 26	infeasible 1.49087	1.70940	[1, 1.70940)
19	IRS	1, 8, 24	1.47641	1.42045	[1, 1.42045)
20	IRS	6, 8, 21	infeasible 1.76204	2.34192	[1, 2.34192)
22	IRS	1, 8, 21	1.20103	1.14416	[1, 1.14416)
27	IRS	1, 25	infeasible 1.86654	7.75194	[1, 7.75194)
28	IRS	6, 8, 26	1.36882	1.34590	[1, 1.34590)

$E_o = \{\text{DMU1, DMU2, DMU3}\}$ where $DMU2 = \frac{2}{3}DMU1 + \frac{1}{30}DMU3$ is in set E′. Multiple optimal lambda solutions are detected in evaluating DMU4 using (1) (see Zhu and Shen, 1995).

First calculate (3), that is

$$\begin{aligned} \varphi_o^* &= \max \varphi_o \\ s.t.\ 0.1\ \lambda_1 + 2\lambda_2 + 40\lambda_3 &\le 3 \\ 0.25\lambda_1 + 2\lambda_2 + 10\lambda_3 &\le 2 \\ 0.1\ \lambda_1 + \lambda_2 + 10\lambda_3 &\ge \varphi_o \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1 \\ \lambda_1, \lambda_2, \lambda_3 &\ge 0 \end{aligned}$$

The optimal value is $\varphi_o^* = 7/6$. Next calculate (10), that is

$$\begin{aligned} (\tau_o^*)^{-1} &= \min \lambda_1 + \lambda_2 + \lambda_3 \\ s.t.\ 0.1\ \lambda_1 + 2\lambda_2 + 40\lambda_3 &\le 3 \times \tfrac{6}{7} = \tfrac{18}{7} \\ 0.25\lambda_1 + 2\lambda_2 + 10\lambda_3 &\le 2 \times \tfrac{6}{7} = \tfrac{12}{7} \\ 0.1\ \lambda_1 + \lambda_2 + 10\lambda_3 &\ge 1 \\ \lambda_1, \lambda_2, \lambda_3 &\ge 0 \end{aligned}$$

We have $\tau_o^* = \frac{210}{1011}$ with $\lambda_1^* = \frac{100}{21}$, $\lambda_2^* = 0$ and $\lambda_3^* = \frac{11}{210}$.

Finally, calculate (11). We have $\sigma_o^* = \frac{70}{52}$ with $\lambda_1^* = 0$, $\lambda_2^* = \frac{5}{7}$ and $\lambda_3^* = \frac{2}{70}$.

Therefore the stability region for the CRS classification is $\left\{\chi\colon \frac{210}{1011} \le \chi \le \frac{70}{52}\right\}$.

Appendix C

We illustrate the RTS sensitivity analysis method on a real world data set consisting of 28 Chinese cities (DMUs) in 1983 from Charnes, Cooper and Li (1989). There are three outputs (gross industrial output value, profit & taxes, and retail sales) and three inputs (labor, working funds, and investment).

Table C.1 reports the results for the 20 inefficient cities. Column 1 gives the DMU numbers which are the same as in Charnes, Cooper and Li (1989). RTS classifications and E_o are reported respectively in columns 2 and 3. Column 4 gives the optimal value to (3), φ_o^*, or the optimal value to (2), ϑ_o^* when (3) is infeasible. τ_o^* or σ_o^* are reported in column 5. The RTS stability regions are reported in the last column.

Two efficient DMUs, namely, DMU23 and DMU28, do not appear in E_o when evaluating other inefficient DMUs. It is easy to see that this data set satisfies RC1, but violates RC2, because $\varphi_o^* > 1$ for DMU2.

Notes

1. The set of DMU_j ($j \in E_o$) may be different for each different DMU_o under evaluation. Furthermore, $E_o \subset BI_o$ for input-orientation (or BO_o for output-orientation) of Seiford and Thrall (1990, p.19) and E_o is related to the Primal Representation Theorem of Charnes, Cooper and Thrall (1991, p. 215) in which a CCR referent group is determined via a strong complementary slackness condition (SCSC) solution. Also, DMUs in E_o may not form an efficient facet (see Thrall, (1996)).
2. We consider the RTS of BCC non-frontier DMUs by their BCC projections.
3. Note that $\varphi_o = \tau_o^* < 1$ is a feasible solution to (3). Therefore $\varphi_o^* \geq \tau_o^*$.

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