Journal of Productivity Analysis, 15, 217–246, 2001

Sensitivity and Stability Analysis in DEA: Some Recent Developments

W. W. COOPER
Red McCombs School of Business, University of Texas at Austin, Austin, Texas 78712-1174, USA

SHANLING LI
Faculty of Management, McGill University, Montreal, Quebec, Canada

L. M. SEIFORD, PH.D.
National Science Foundation, 4201 Wilson Blvd., Arlington, Virginia 22230, USA

KAORU TONE
National Graduate Institute for Policy Studies, 2-2 Wakamatsu-cho, Shinjuku-ku, Tokyo 162-8677, Japan

R. M. THRALL
Jones Graduate School of Management, Rice University, Houston, Texas 77204-1892, USA

J. ZHU
Department of Management, Worcester Polytechnic Institute, Worcester, Massachusetts, 01609, USA

Abstract

This paper surveys recently developed analytical methods for studying the sensitivity of DEA results to variations in the data. The focus is on the stability of classification of DMUs (Decision Making Units) into efficient and inefficient performers. Early work on this topic concentrated on developing solution methods and algorithms for conducting such analyses after it was noted that standard approaches for conducting sensitivity analyses in linear programming could not be used in DEA. However, some of the recent work we cover has bypassed the need for such algorithms. Evolving from early work that was confined to studying data variations in only one input or output for only one DMU at a time, the newer methods described in this paper make it possible to determine ranges within which all data may be varied for *any* DMU before a reclassification from efficient to inefficient status (or *vice versa*) occurs. Other coverage involves recent extensions which include methods for determining ranges of data variation that can be allowed when all data are varied simultaneously for *all* DMUs. An initial section delimits the topics to be covered. A final section suggests topics for further research.

Keywords: Efficiency, Data Variations, Sensitivity, Stability

1. Introduction

This paper surveys recently developed approaches for determining the sensitivity (or stability) of results secured in DEA analyses. The emphasis is on "methodological" rather than "substantive" approaches—where the latter take the form of general statements about the stability of DEA results *per se*, as in , for instance, the early book on DEA by Sexton, Silkman and Hogan (1986).

We proceed in the spirit of Grosskopf (1996) who, in her survey, distinguishes between (1) the statistical (or stochastic) approaches which are commonly used in the economics and econometrics literatures, including the literature treating frontier estimation and efficiency evaluation, and (2) the deterministic approaches like those commonly used in the literatures of operations research and management science. The discussion in Grosskopf is oriented toward (1). Ours is oriented toward (2). In particular, the meaning we assign to the terms "sensitivity" (or stability) analysis conforms to the usages commonly assigned to these terms in the literature of mathematical programming.

We confine attention to analyses of effects occurring from data variations. Other topics such as sensitivity to model changes or to diminution or augmentation of the number of DMUs, e.g., as in the sampling distributions discussed in Simar and Wilson (1998), are not examined in detail. An important list of papers on the topic we address is to be found in a long list of publications emanating from research jointly undertaken by Charnes and Neralic. See the references cited in Seiford (1994). This work was originally confined to stability of DEA results under data variations for one input or one output. This was subsequently extended to allow simultaneous variations of all data—first under proportional variations of all data in Charnes and Neralic (1992b) and subsequently extended to arbitrary changes in all data for the "additive model," as reported in Neralic (1997).

Tracking this line of work by Charnes, Neralic, *et al.* would involve tracing its origins back to Charnes, Cooper, Lewin, Morey and Rousseau (1985) which, in turn, represented a response to their finding that the sensitivity analysis methods used in linear programming were not suited for use in DEA. This means, as noted in Charnes *et al.* (1985), that new algorithms were needed because the data to be varied occur on both sides of the constraints in the linear programming models used in DEA. Fortunately a good start could be made in developing such algorithms because preparatory work was available from preceding literature in the form of the publication by Charnes and Cooper (1968) which deals with the effects of data variations on matrix inverses like those used for sensitivity analysis in linear programming. Theorems in the latter paper could therefore provide underpinning for extensions that were needed (and have since been effected) for use with DEA.

The emphasis on new and needed algorithms leads us to characterize this line of work as "algorithmically oriented." This is important, of course, but an effort to discuss this work in the detail needed to do it justice would greatly lengthen the present paper. Because we do not treat this work in detail in the present paper we direct interested readers to the extensive references provided in Neralic (1997). However, we do follow the precedents of Charnes and Neralic by turning attention to (i) sensitivity analysis of inputs and outputs in one DMU and then (ii) extend this to concepts and methods that have been developed for treating simultaneous variations in all inputs and all outputs in all DMUs. For (i) we examine the introduction of metric concepts by Charnes *et al.* (1992a, 1996) which are applied in ways that make it possible to determine allowable variations in all inputs and outputs for one DMU. For (ii) we first turn to work by Thompson *et al.* (1994, 1996) which uses "multiplier" model approaches to data variations in all inputs and all outputs in all

DMUs simulataneously. We then follow this by describing treatments of this same topic via "envelopment" models as in Seiford and Zhu (1998b, 1998c). See also Neralic (2000).

All of this work is confined to determining the stability of the originally obtained classifications of efficient into inefficient DMUs. Topics such as the magnitude of changes in inefficiencies (in each input and output) are covered only as a byproduct of these efficiency-inefficiency reclassifications. Hence a final section suggests this and other topics for further research.

New contributions to the thus summarized developments are introduced at various points which include new theorems as well as interpretations and suggestions for use. However, this is done only to clarify the discussion or to repair omissions. In any case we restrict the discussion in this paper to analytically formulated (mathematical) methods for examining stability and sensitivity of results to data variations with given variables, given DMUs, and given criteria for evaluating efficiency. That is, we do not cover the now numerous stability analyses which have been conducted in simulation studies like those initiated by Banker *et al.* (1988)—such simulations have been subsequently extended to cover changes in the variables used by the DMUs to be considered, as in, for instance, Banker *et al.* (1996) or Ahn and Seiford (1993). We now conclude this introductory discussion with the following comment which is taken from Cooper, Seiford and Tone (1999, p. 252):

> *Comment*: As in statistics or other empirically oriented methodologies, there is a problem involving degrees of freedom, which is compounded in DEA because of its orientation to *relative* efficiency. In the "envelopment model," the number of degrees of freedom will increase with the number of DMUs and decrease with the number of inputs and outputs. A rough rule of thumb which can provide guidance is to choose a value of n that satisfies
>
> $$n \geq \max\{m \times s, 3(m + s)\}$$
>
> where n = number of DMUs, m = number of inputs and s = number of outputs. Hereafter we assume that this (or other) degrees of freedom conditions are satisfied and that there is no trouble from this quarter.

We hereafter assume the absence of problems that might arise from such degrees-of-freedom considerations.

2. Definition of Efficiency

Various definitions of efficiency are available in the DEA literature. Unless otherwise noted, the one we use is referred to as the "Pareto-Koopmans" definition of efficiency which we articulate as follows,

Definition 1 (Pareto-Koopmans Efficiency). A "Decision Making Unit" is efficient if and only if it is not possible to improve some of its inputs or outputs without worsening some of its other inputs or outputs.

The term "Decision Making Unit," which we abbreviate to "DMU," refers to the entity (school, hospital, business firm, etc.) which is regarded as being responsible for converting inputs into outputs. The above definition is then equivalent to asserting that a DMU is efficient if and only if it is not dominated by some other DMU (or combination of DMUs) with which it can be compared.

This orientation means that we are restricting attention to "technical" aspects of efficiency (sometimes referred to as "waste"). Evaluations arising from prices, costs or preferences are not addressed in this paper. No substitutions, exchanges or further utilization of resources is needed to eliminate these inefficiencies when dominance of one DMU over another DMU is present. In this sense the approach employed is "value free" and "objective"—i.e., given the choice of inputs, outputs and DMUs, the same inefficiency vs. efficiency characterizations will be secured for all users applying the same DEA model to the same data. See Post (1999) for ways to incorporate utilities and preferences into DEA. See also Joro, Korhonen and Wallenius (1998) for operationally implementable ways to identify decision maker preferences in a manner that makes it possible to incorporate them in the evaluations that will be effected.

The analyses in DEA are "data based." That is, the models used are non-parametric and are therefore relatively free of the restrictive assumptions employed with other approaches. There are, however, alternative models that can be used to identify differences such as "mix" vs. "purely technical" inefficiency—where the latter, but not the former, does not involve any change in input proportions used, or output proportions produced. The model we start with is an extension of the "additive model," as employed in Charnes *et al.* (1992a, 1996), which bypasses distinctions in mix and purely technical inefficiency and simply identifies all inefficiencies in terms of non-zero slacks. (See Cooper, Park and Pastor (1999) for further discussion of "mix" vs. "purely technical" and other types of inefficiencies, and see Ahn, Charnes and Cooper (1988) for a discussion of the different DEA models and their relations to each other.)

3. Metric Approaches for Inefficient DMUs

Having identified efficient and inefficient DMUs in a DEA analysis, one may want to know how sensitive these identifications are to possible variations in the data. A new avenue for sensitivity analysis was opened by Charnes *et al.* (1992a).[1] The basic idea is to use concepts such as "distance" or "norm" (= length of a vector), as defined in the mathematical literature dealing with metric spaces, and use these concepts to determine "radii of stability" within which data variations will not alter a DMU's classification from efficient to inefficient status (or *vice versa*). See Charnes and Cooper (1961, Appendix A) for a discussion of these and other metric concepts in which the usual mathematical treatments are simplified by translating them into algebraic form for use in linear programming.

The classifications obtained from these "radii of stability" can range from "unstable" to "stable" with the former being identified with an infinitesimal radius and latter being identified by a radius of some finite value within which reclassification does not occur. A point like F in Figure 1 provides an example identified as stable. A point like A, however, is unstable because an infinitesimal perturbation to the left of its present position would alter its status from inefficient to efficient.

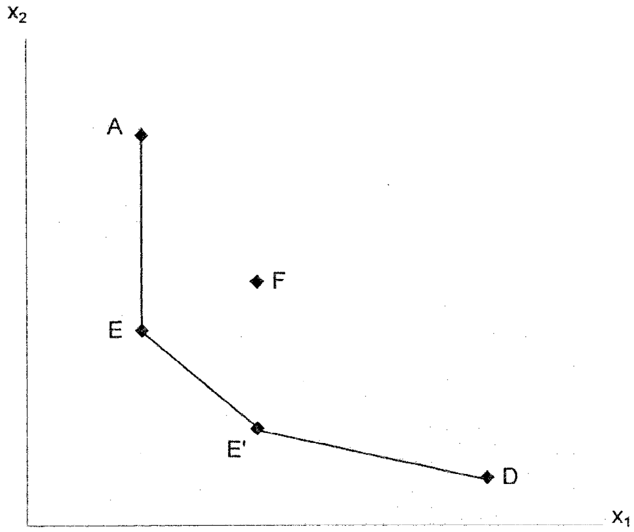

Figure 1. Stable and unstable DMU.

Among the metrics examined by Charnes *et al.* (1992a), we select only the Chebychev ($= l_\infty$) norm and use it to portray the essentials in these developments. As in Charnes *et al.* (1992a) we use the following model to give form to these ideas,

$$\max \delta$$

subject to

$$y_{ro} = \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ - \delta d_r^+, \qquad r = 1, \ldots, s$$

$$x_{io} = \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- + \delta d_i^-, \qquad i = 1, \ldots, m$$

$$1 = \sum_{j=1}^{n} \lambda_j$$

$$0 \leq \delta, \lambda_j, s_r^+, s_i^-, \quad \forall i, j, r. \tag{1}$$

Here $y_{rj}, x_{ij} \geq 0$ represent $r = 1, \ldots, s$ outputs and $i = 1, \ldots, m$ inputs for DMU_j, $j = 1, \ldots, n$, and y_{ro}, x_{io} represent the observed value of output r and input i for DMU_o, the DMU to be evaluated relative to all of the other DMUs (including $\text{DMU}_j = \text{DMU}_o$).

All variables, including δ, are constrained to be nonnegative in (1) while the d_r^+ and d_i^- are prescribed as positive constants (or weights). To simplify the discussions that follow we now assume that these weights are all unity so that, with all $d_i^- = d_r^+ = 1$, the solution to (1) may be written

$$\sum_{j=1}^{n} y_{rj}\lambda_j^* - s_r^{+^*} = y_{ro} + \delta^*, \qquad r = 1, \ldots, s$$

$$\sum_{j=1}^{n} x_{ij}\lambda_j^* + s_i^{-^*} = x_{io} - \delta^*, \qquad i = 1, \ldots, m \tag{1.1}$$

where "*" indicates an optimum and the value of δ^* represents the maximum that this model allows consistent with the solution on the left.

The above formulations are for an *inefficient* DMU, which continues to be inefficient for all data alterations which yield improvements from y_{ro} to $y_{ro} + \delta^*$ and from x_{io} to $x_{io} - \delta^*$. This means that no reclassification to efficient status will occur within the open set defined by the value of $\delta^* > 0$. Referred to as a "radius of stability," this value of δ^* defines a symmetric region within which all inputs and outputs for DMU_o can be improved without producing a change from inefficient to efficient status.

We now alter the above formulations to

$$\max \delta + \varepsilon \left(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^- \right)$$

subject to

$$y_{ro} = \sum_{j=1}^{n} y_{rj}\lambda_j - \delta - s_r^+ \qquad r = 1, 2, \ldots, s;$$

$$x_{io} = \sum_{j=1}^{n} x_{ij}\lambda_j + \delta + s_i^- \qquad i = 1, 2, \ldots, m;$$

$$1 = \sum_{j=1}^{n} \lambda_j$$

$$0 \le \delta, \lambda_j, s_i^-, s_r^+, \quad \forall i, j, r. \tag{2}$$

In this case (as previously noted) we have set $d_i^-, d_r^+ = 1$, $\forall i, j, r$ in order to simplify matters. See the discussion for (5), below. We have also modified the models in Charnes *et al.* (1992a, 1996) by incorporating the slacks s_r^+ and s_i^- in the objective where they are multiplied by $\varepsilon > 0$, a non-Archimedean element defined to be smaller than any positive real number.

It is not necessary to explicitly assign a value to $\varepsilon > 0$. Instead, as in most DEA computer codes, this is taken care of operationally by using a two-stage computation which may be formalized as follows. Stage 1 secures an optimal $\delta = \delta^*$ without reference to the non-zero slack possibilities. The latter are dealt with in stage 2 by incorporating this value of δ^* in the following model

$$\max \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-$$

subject to

$$y_{ro} + \delta^* = \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ \qquad r = 1, 2, \ldots, s;$$

$$x_{io} - \delta^* = \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- \qquad i = 1, 2, \ldots, m;$$

$$1 = \sum_{j=1}^{n} \lambda_j$$

$$0 \leq \lambda_j, s_i^-, s_r^+; \quad \forall i, j, r. \tag{3}$$

Thus no exchange between δ^* and the slack values is permitted. This reflects the fact that these slack values are multiplied by $\varepsilon > 0$ in the objective of (2) and the definition of ε makes it disadvantageous to effect increments in the slacks, however large, in exchange for decrements in δ^*, however small.

Figure 2, below, helps to portray what is happening by reference to the square surrounding F. This square (a symmetric figure referred to as a "unit ball" in Charnes *et al.*) is generated from the vector with length δ^* indicated by the arrow. This length is determined by the point of intersection with the frontier. That is, this point of intersection determines the radius of stability because $x - \delta^*$, $y + \delta^*$, as in (1.1), provides contact with a point where a change is effected from inefficient to efficient status.

For F the radius of stability is determined by a point of intersection with the efficient frontier where the term efficiency refers to Pareto-Koopmans efficiency as stated in Definition 1

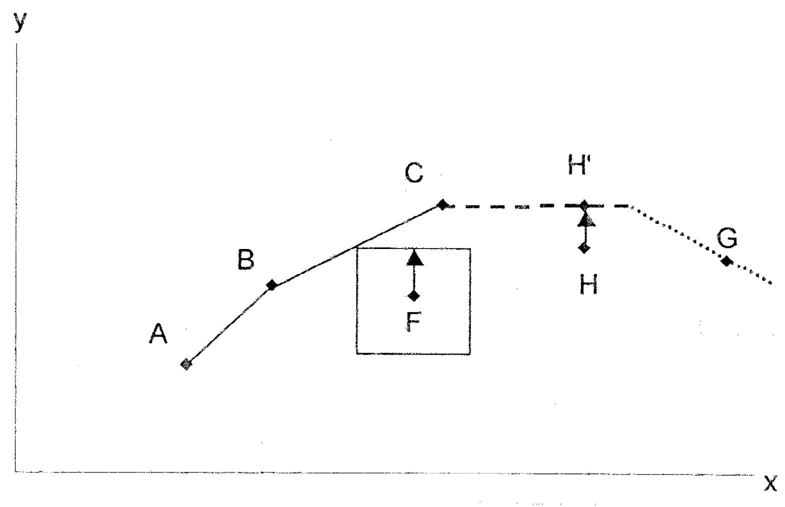

Figure 2. A radius of stability.

in the preceding section. This is a consequence of the 2-stage optimization used in (2) and (3). When (1) is used, however, the radius of stability may be determined by a frontier point which is only “weakly efficient.” A case in point is exhibited by H in Figure 2 which has its radius of stability determined by H′ which is a point on the frontier that admits reductions in the input amount x without reducing the output amount y.

4. Relations to Other Models

We develop these latter comments in more detail by examining how these developments relate to other models. We start with an additive model. Introduced into the DEA literature by Charnes *et al.* (1985), this version of an additive model is selected in order to align it with the analogous formulations in (1) and (2).

$$\max \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-$$

subject to

$$\begin{aligned}
&y_{ro} = \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+, \qquad r = 1, 2, \ldots, s;\\
&x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^-, \qquad i = 1, 2, \ldots, m;\\
&1 = \sum_{j=1}^{n} \lambda_j\\
&0 \le \lambda_j, s_i^-, s_r^+; \quad \forall i, j, r.
\end{aligned} \tag{4}$$

This model may be used to determine the efficiency of any DMU_o in the sense of Definition 1 by reference to

Definition 2 (Efficiency). DMU_o, the DMU being evaluated, is efficient if and only if an optimum is attained with all slacks zero in (4).

As can be seen, this model differs from (1) and (2) only in the statement of its objective and omission of the extra variable, $\delta \ge 0$, around which the objective in (1) and (2) is oriented. However, the two models are members of the same family in a manner that allows us to use (4) in analyzing properties of model (1) and (2). We show this by means of the following

THEOREM 1 *A solution to (2) yields values*

$$\begin{aligned}
y_{ro}^* &= y_{ro} + \delta^* + s_r^{+^*}, \qquad r = 1, 2, \ldots, s;\\
x_{io}^* &= x_{io} - \delta^* - s_i^{-^*}, \qquad i = 1, 2, \ldots, m;
\end{aligned}$$

in which y_{ro}^*, x_{io}^* are the coordinates of an efficient point.

Proof. Using the thus defined values of y^*_{ro}, x^*_{io}, we utilize (4) and apply Definition 2 to prove this theorem as follows,

$$\max \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-$$

subject to

$$y^*_{ro} = \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ \qquad r = 1, 2, \ldots, s;$$

$$x^*_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- \qquad i = 1, 2, \ldots, m;$$

$$1 = \sum_{j=1}^{n} \lambda_j$$

with all variables constrained to be non-negative. Suppose we have an optimum solution to this problem which we can write as

$$\begin{aligned} y^*_{ro} &= \sum_{j=1}^{n} y_{rj}\hat{\lambda}_j - \hat{s}_r^+, \qquad r = 1, 2, \ldots, s; \\ x^*_{io} &= \sum_{j=1}^{n} x_{ij}\hat{\lambda}_j + \hat{s}_i^-, \qquad i = 1, 2, \ldots, m; \\ 1 &= \sum_{j=1}^{n} \hat{\lambda}_j \end{aligned}$$

Rewriting this solution we have

$$\begin{aligned} y_{ro} &= \sum_{j=1}^{n} y_{rj}\hat{\lambda}_j - \hat{s}_r^+ - \delta^* - s_r^{+^*}, \qquad r = 1, 2, \ldots, s; \\ x_{io} &= \sum_{j=1}^{n} x_{ij}\hat{\lambda}_j + \hat{s}_i^- + \delta^* + s_i^{-^*}, \qquad i = 1, 2, \ldots, m; \\ 1 &= \sum_{j=1}^{n} \hat{\lambda}_j. \end{aligned}$$

We thus have a solution to (2) with the same δ^* but with new slacks $(\hat{s}_r^+ + s_r^{+^*})$ and $(\hat{s}_i^- + s_i^{-^*})$. If any of these $\hat{s}_r^+$, $\hat{s}_i^-$ values were positive we would be contradicting the assumption that the $s_r^{+^*}$, $s_i^{-^*}$ maximized the sum represented in the objective of the second stage optimization for (2) that is represented in (3). This contradiction can be avoided only if all of the values $\hat{s}_r^+$, $\hat{s}_i^-$ are zero. It follows from Definition 2 that these y^*_{ro}, x^*_{io} are the coordinates of an efficient point. ■

As a consequence of this theorem we can regard the "stability oriented" models developed by Charnes *et al.* (1992, 1996) as extensions of "efficiency oriented" versions of additive models. In fact, the desired efficiency evaluation orientation is obtained by using the $\delta^* + s_r^{+^*}$ and $\delta^* + s_i^{-^*}$ to define new slacks for use in the corresponding additive model. Hence we find that this extension of the additive model provides both estimates of inefficiency in each input and output and, simultaneously, provides a measure of the stability of these estimates.

These $\delta^* \geq 0$ also play a role analogous to the optimal values of radial measures in models which use radial measures of efficiency. They differ from the radial measure models, however, because the δ^* values appear in both input and output constraints whereas the radial measures apply only to one constraint set or the other, in mutually exclusive fashion.

We can gain still further insight by extending the analysis to efficient DMUs via the following theorem.

THEOREM 2 DMU_o *is (Pareto-Koopmans) efficient if and only if both* δ^* *and all slacks are zero in (2).*

Proof. Suppose the conditions specified in the theorem hold. If $\delta^* = 0$ in stage 1 of (2) then (3) and (4) are equivalent. Hence if all slacks are zero in (3) they will also be zero in (4) in which case DMU_o is efficient by Definition 2. Now suppose DMU_o is efficient. It follows that all slacks and δ^* must be zero in (3). For, if this were not the case, then all slacks at zero could not be optimal for DMU_o in (4). The latter outcome, however, would contradict the efficient status assumed for DMU_o. ■

THEOREM 3 *Some slacks must be zero in an optimum for (3).*

Proof. Suppose an optimum could be secured with all slacks positive in (3). Then we could set $\Delta^* = \min\{s_i^{-^*}, s_r^{+^*} \mid i = 1, \ldots, m; r = 1, \ldots, s\} > 0$. Subtracting $\Delta^* > 0$ from all slacks yields a new set of non-negative slacks with at least one equal to zero. Adding the thus subtracted Δ^* to the previous value yields a solution to (3) $\delta^* + \Delta^* > \delta^*$ and this contradicts the hypothesis that δ^* was optimal for the stage 1 use of (2). ■

COROLLARY 1 *If* $\delta^* = 0$ *in (2) and some slacks are not zero in (3) then* DMU_o *is weakly efficient.*

COROLLARY 2 *If* $\delta^* = 0$ *in (1) then* DMU_o *is at least weakly efficient.*

Remark. We might note that the additive models (unlike radial models) do not distinguish between weak efficiency and technical inefficiency. Here, too, however, an extension is effected as is made clear by theorem 2 and corollary 1.

We can now sharpen these results by noting that $\delta^* = 0$ identifies DMU_o as being on a boundary of the production possibility set because this means that no change in data is needed to move DMU_o to efficient (or weakly efficient) status. However, the converse is not true. To see that this is so refer to the point G in Figure 2 which is on the boundary of the production possibility set where congestion occurs.[2] However, a non-zero radius of stability at least as large as for H is needed before a change from inefficient to efficient status will occur for G. This follows because, via Theorem 1, we know that C will be the efficient point used to evaluate H when (2) is used.

We conclude this part of our discussion of this approach to evaluating the stability of results for inefficient DMUs with the following comment.

Comment. For further insight into the meaning of δ^* in these various uses, we return to the weighting system in (1) and use this information to reformulate (2) in the following manner

$$\max \delta + \varepsilon \left(\sum_{r=1}^{s} s_r^+/d_r^+ + \sum_{i=1}^{m} s_i^-/d_i^- \right)$$

subject to

$$\delta = \frac{-y_{ro}}{d_r^+} + \sum_{j=1}^{n} \frac{y_{rj}\lambda_j}{d_r^+} - \frac{s_r^+}{d_r^+}, \qquad r = 1, \ldots, s$$

$$\delta = \frac{x_{io}}{d_i^-} - \sum_{j=1}^{n} \frac{x_{ij}\lambda_j}{d_i^-} - \frac{s_i^-}{d_i^-}, \qquad i = 1, \ldots, m$$

$$1 = \sum_{j=1}^{n} \lambda_j, \tag{5}$$

with all variables constrained to be non-negative. As is evident from this formulation, the δ^* values are stated as ratios relative to the positive weights used for each constraint. Hence δ^* is a “dimension free” ratio: The weights are thus to be stated in the units that are pertinent to each constraint. The preceding analysis in this part of our paper simplified matters by assuming a special case—*viz.*, that these weights all had unity values. Nevertheless our theorems and interpretations continue to apply when this equal weight assumption is dropped—although they will need to be stated in more complex manners for this general formulation. This can be made clear by assuming that (5) is stated in terms of new data and new variables defined as follows,

$$\hat{y}_{rj} = \frac{y_{rj}}{d_r^+}, \qquad \hat{x}_{ij} = \frac{x_{ij}}{d_i^-}$$

$$\hat{s}_r^+ = \frac{s_r^+}{d_r^+}, \qquad \hat{s}_i^- = \frac{s_i^-}{d_i^-}$$

for $i = 1, \ldots, m$; $r = 1, \ldots, s$; $j = 1, \ldots, n$. The thus defined new variables and data could then be used in the above theorems.

5. Metric Approaches for Efficient DMUs

The above model is directed to cases where “improvements” can be effected in the data for an inefficient DMU until it changes to an efficient performer. We now turn in the opposite direction and examine the "worsenings" needed in the inputs and outputs of an efficient DMU that will cause data changes sufficient to characterize it as an inefficient performer.

For this purpose Charnes *et al.* (1996) reverse the signs associated with δ in (1), and also reverse the objective as is done in the following model,

$$\min \delta$$

subject to

$$y_{ro} = \sum_{j=1, j\neq 0}^{n} y_{rj}\lambda_j - s_r^+ + \delta, \qquad r = 1, \ldots, s$$

$$x_{io} = \sum_{j=1, j\neq 0}^{n} x_{ij}\lambda_j + s_i^- - \delta, \qquad i = 1, \ldots, m$$

$$1 = \sum_{j=1, j\neq 0}^{n} \lambda_j \tag{6}$$

Here, again, all variables are constrained to be nonnegative. All variables and data are defined as before but in this case $j \neq o$ refers to the omission of the efficient DMU_o which is being analyzed. This is needed because without this omission the result will always be unstable as defined by Charnes *et al.*, (1992a, 1996)[3] *viz.*,

Definition 3. The coordinates of the point associated with an efficient DMU will always have both efficient and inefficient points within a radius of $\varepsilon > 0$, however small the value of ε. Any point with this property is "unstable."

This property, we may note, is not confined to points associated with efficient DMUs. For instance, as previously noted, point A in Figure 1 has this property since a slight variation to the left will change its status from inefficient to efficient. In any case, a solution, δ^*, provides a radius in Chebyshev norm with a value that is to be minimally attained before an efficient DMU is changed to inefficient status.

Remark. In this case we are again reverting to the assumption that $d_r^+ = d_i^- = 1$; $\forall i, r$ for the formulation used by Charnes et al. (1992a, 1996). Slack variables introduced into the objective when moving from (1) to (2) are omitted because, by assumption, we are dealing with efficient points.

To see what is involved we can assume that B in Figure 2 is the efficient point to be considered. Removal of B as required for (6) would result in a new frontier taking form with the line segment connecting A and C. δ^* would then represent the radius of stability with its value determined by the Chebyshev norm that is being used for this purpose. This, in turn, would be associated with the square (or unit ball) that is used to represent this norm in the manner we previously discussed for F.

As can be seen, there are additional questions that flow from this analysis. For instance, removal of B in this fashion would evidently affect the stability radius for F as well as other points. This raises a question as to how such further effects are to be treated in a stability analysis. Later in this paper this topic is dealt with in a manner that resolves the issue of how to select a DMU for analysis. Here we only need to note that Charnes *et al.* (1992a, 1996) recommend using the above formulations with their associated ℓ_∞ metric only as the start for a sensitivity analysis. They then go on to study other metrics which can provide additional results.

6. Solvability

We now move to issues of "solvability" which arise with the removal of DMU_o, the efficient DMU for which the above stability analysis is designed. As noted earlier in this paper, Andersen and Petersen (1993) also adopt an approach which omits the DMU_o under analysis in a manner analogous to (3) which they use for ranking DMUs by reference to a property that they refer to as "superefficiency." Unlike the Andersen and Petersen approach, the Charnes *et al.* (1992a, 1996) formulations do not encounter issues of solvability when such a DMU_o omission is made. (See Thrall (1996) and Zhu (1999) for detailed treatments of this topic of solvability.) The Charnes *et al.* (1992a, 1996) approach always has a minimizing solution which identifies a closest efficient point in the reduced set of solutions (possibly vacuous) which remains after DMU_o is deleted.

The need for a proof of this last statement was made apparent by comments elicited during the presentation of this paper in the 6th European Workshop on Efficiency and Productivity so we proceed to the following demonstration: We first restate (6) in the following equivalent form

$$
\begin{aligned}
&\min \delta \\
&\text{subject to} \\
&\delta \geq y_{ro} - \sum_{j=1, j\neq o}^{n} y_{rj}\lambda_j, \qquad r = 1, \dots, s \\
&\delta \geq -x_{io} + \sum_{j=1, j\neq o}^{n} x_{ij}\lambda_j, \qquad i = 1 \dots, m \\
&1 = \sum_{j=1, j\neq o}^{n} \lambda_j,
\end{aligned} \tag{7}
$$

where all variables are constrained to be non-negative. The solution to this problem may be formulated in "min max" terms as follows,

$$
\begin{aligned}
\delta^* = \min_{\lambda\geq 0} \max_{i,r} \Bigg\{ &\Bigg\{ y_{ro} - \sum_{j=1, j\neq o}^{n} y_{rj}\lambda_j \mid r = 1, \dots s \Bigg\}, \\
&\Bigg\{ \sum_{j=1, j\neq o}^{n} x_{ij}\lambda_j - x_{io} \mid i = 1, \dots, m \Bigg\}, 0 \Bigg\}
\end{aligned} \tag{8}
$$

with all variables non-negative and $\sum_{j=1, j\neq o}^{n} \lambda_j = 1$.

Such a solution always exists. To see that this is so, we rewrite (6) in the following form,

$$\min \delta$$

subject to

$$\delta - s_r^+ = y_{ro} - \sum_{j=1,\, j\neq o}^{n} y_{rj}\lambda_j, \qquad r = 1, \ldots, s$$

$$\delta - s_i^- = -x_{io} + \sum_{j=1,\, j\neq o}^{n} x_{ij}\lambda_j, \qquad i = 1, \ldots, m$$

$$1 = \sum_{j=1,\, j\neq o}^{n} \lambda_j$$

$$0 \leq \delta, \lambda_j, s_i^-, s_r^+ \quad \forall i, j, r. \tag{9}$$

The slacks and δ are always of opposite sign in the constraints. Hence we can always obtain a solution to this problem in the following manner. If any of the first set of $m+s$ constraints on the right is positive, we set δ equal to the maximum of these values and use the slacks to attain the equalities required in (6). If none of the expressions on the right are positive we set $\delta = 0$ and again use the slacks to obtain the required equalities.

From this development we see that (6) always has a solution. Although unbounded above, the values of the slacks and δ are bounded below so no trouble is encountered in the direction toward which the objective in (6) is oriented. The λ values are bounded in all directions by the conditions $\sum_{j=1,\, j\neq o}^{n} \lambda_j = 1, \lambda_j \geq 0\ \forall j, j \neq o$. Hence troubles from this quarter are not encountered with either the max or min operators in the preceding expressions. We have therefore proved the following

THEOREM 4 *The problem (6) always has a finite optimum value which establishes a radius of stability.*

Remark. The above approach also establishes a new way of ranking DMUs by reference to their radii of stability. Unlike Andersen and Petersen (1993) it does not encounter problems of solvability. See Thrall (1996) and Zhu (1999). This principle of ranking can also be applied to inefficient DMUs (in reverse order) and this extension is also not available in the Andersen-Petersen approach since "amounts" of inefficiency may be coming from different facets and hence involve comparisons with the different peer groups used to generate these facets. See Charnes *et al.* (1989) on the need for explicitly specifying the principle on which a ranking in DEA may be based and see Dula and Hickman (1997) for additional reasons why the Andersen-Petersen approach cannot be used for ranking.

7. Multiplier Model Approaches

There is a potential for interactions with other δ^* values when radii of stability for efficient points are being determined. Recall, for instance, the effect on the δ^* for F in Figure 2 that was noted in our discussion of the way the radius of stability for B was to be determined from (6). The issue of how to determine which DMUs should be of interest in such analyses also arises. Both of these topics are addressed in the approach we now describe which uses a different class of models and relies on "multiplier model" variables which are dual to the "envelopment model" variables used in the metric approaches we have been discussing.

The approach to which we now turn allows all data for every DMU to be varied simultaneously. Pioneered by Thompson, Dharmapala and Thrall (1994) this method of stability analysis was developed further in Thompson *et al.* (1996) from which we draw the following dual pair of linear programming problems,

Envelopment Model	Multiplier Model	
minimize$_{\theta,\lambda}$ θ	maximize$_{u,v}$ $z = uy_o$	
subject to	subject to	
$Y\lambda \geq y_o$	$u \geq 0$	(10)
$\theta x_o - X\lambda \geq 0$	$v \geq 0$	
$\lambda \geq 0$	$uY - vX \leq 0$	
θ unrestricted	$vx_o = 1.$	

Here Y, X and y_0, x_o are data matrices and vectors of outputs and inputs, respectively, and λ, u, v are vectors of variables (λ: a column vector; u and v: row vectors). θ, a scalar, which can be positive, negative or zero in the envelopment model is the source of the condition $vx_o = 1$ which appears at the bottom of the multiplier model.

No allowance for nonzero slacks is made in the objective of the envelopment model. Thus the *positivity* requirement associated with the commonly used non-Archimedean element, ε, is absent from both members of this dual pair. Thompson *et al.* refer to Charnes, Cooper and Thrall (1991) to justify the omission of this non-Archimedean element. For present purposes, however, we only need to note that the sensitivity analyses we will now be considering are centered around the set, E, of efficient extreme points and these points always have a unique optimum with nonzero slack solutions for the envelopment (but not the multiplier) model. See Charnes, Cooper, Thrall (1986, 1991).

In contrast to the use of additive models in Charnes *et al.* (1992a, 1996), we now turn to the radial models, as in the problem on the left in (10). Also in contrast to the previous approach the analysis used by Thompson *et al.* (1994, 1996) is carried forward via the multiplier models on the right in (10). This makes it possible to exploit the fact that the values u^*, v^* which are optimal for the DMU being evaluated will remain valid over some (generally positive) range of variation in the data.[4]

Thompson *et al.* (1994, 1996) exploit the latter property by defining a new vector $w = (u, v)$ which they use to define a function $h_j(w)$ as follows,

$$h_j(w) = \frac{f_j(w)}{g_j(w)} = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}. \tag{11}$$

Next, let

$$h_o(w) = \max_{j=1,\dots,n} h_j(w) \tag{12}$$

so that

$$h_o(w) \geq h_j(w), \quad \forall j. \tag{13}$$

We now note that (11) returns matters to the nonlinear version of the CCR *ratio* form as introduced in Charnes, Cooper and Rhodes (1978). Hence, we need not be concerned with continued satisfaction of the condition $vx_o = 1$ in (10) when we begin to study variations in the data since that condition emerges only after the ratio form is transformed to its linear programming equivalent.

When an optimal w^* does not satisfy (13), the DMU_o being evaluated is said to be "radial inefficient." The term is appropriate because this means that $\theta^* < 1$ will occur in the envelopment model. The full panoply of relations between the CCR ratio, multiplier and envelopment models is thus brought into play without any need for extensive recomputations.

Among the frontier points (for which $\theta^* = 1$), attention is directed by Thompson *et al.* (1994, 1996) to "extreme efficient points." In particular, attention is centered on points in the set which are referred to as E (= extreme efficient) in Charnes, Cooper and Thrall (1986, 1991). For points in this set we have

$$h_o(w^*) > h_j(w^*) \quad \forall j \neq o, \tag{14}$$

for some multiplier w^*. That is, DMU_o will be extreme efficient if and only if there exists a vector w^* for which this strict inquality holds.

How this w^* is to be selected will be discussed below. Here we only note that this (strict) inequality will generally remain valid over some range of variation in the data. Thus, the remarks made when introducing (11) are intended to apply to this strict inequality. In more detail, the strict inequality in (14) assumes the form

$$h_o(w^*) = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{i=1}^{m} v_i^* x_{io}} > \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}} = h_j(w^*) \quad \forall j \neq o, \tag{15}$$

which means that DMU_o is more efficient than any other DMU_j and hence will be rated as fully efficient by DEA, and will remain so over some range of data variation.

Thompson, *et al.* (1994, 1996) employ a ranking principle which they formulate as:

> "If DMU_o is more efficient than all of the other DMU_j relative to the vector w^*, then DMU_o is said to be top ranked."

Advantage is taken of the properties noted for (14) and (15) by holding w^* fixed while the data are varied. DMU_o is then said to continue to be "top ranked" as long as (14) and (15) continue to hold.

Thompson *et al.* (1996) carry out experiments in which the data are allowed to vary in different ways. Among these possibilities we examine only the following one: the outputs will all be decreased and the inputs will all be increased by a stipulated amount (or percentage). This same treatment is accorded to all of the DMUs which are efficient (including those which are not extreme efficient). For the other DMU_j (which are all inefficient) the reverse adjustment is made: All outputs are increased and all inputs are decreased. In this way the value of the ratio in (15) will be decreased for both DMU_o in

Table 1. Data for a sensitivity analysis.

	E-Efficient*			Not Efficient		
DMU	1	2	3	4	5	6
Output: y	1	1	1	1	1	1
Input: x1	4	2	1	2	3	4
Input: x2	1	2	4	3	2	4

*E-Efficient = Extreme Point Efficient

Table 2. Initial solutions.

	DMU1	DMU2	DMU3
DMU	$h_j(w^1)$	$h_j(w^2)$	$h_j(w^3)$
1	1.000	0.800	0.400
2	0.714	1.000	0.714
3	0.400	0.800	1.000
4	0.500	0.800	0.667
5	0.667	0.800	0.550
6	0.357	0.500	0.357

(10) and for the other efficient DMUs while the ratios for the other DMU_j will be increased. Continuing in this manner a reversal can be expected to occur at some point in (14)—in which case DMU_o will no longer be "top ranked,"—which means that it will then lose the status of being fully (or extreme) DEA efficient.

Table 1 taken from Thompson *et al.* (1994) will be used to illustrate the procedure in a simple manner by varying the data only for the inputs x_1, x_2.

Table 2 records the initial solutions obtained by applying the multiplier model in (10) to each of DMU_1, DMU_2 and DMU_3. In conformance with (14) and (15), these values show that all 3 of these DMUs are extreme efficient. (See the remark made immediately after (14)). These solutions show DMU_1, DMU_2 and DMU_3 to be top ranked in their respective columns. See the remark following (14).

The fact that gaps occur between the top and other ranks, as reflected in Table 2, suggests that some range of data variation can be undertaken without changing this top-ranked status in any of these three columns. To validate this last statement we follow Thompson *et al.* (1996) and hold $w^* = (u^*, v^*)$ fixed while we introduce 5% increases in each of x_1 and x_2 for DMU_1, DMU_2 and DMU_3. Simultaneously, we decrease these inputs by 5% for the other (inefficient) DMUs.

This produces the results shown in Table 3. With these variations in the coefficients of w^*, changes in the $h_j(w^*)$ values will occur, as can be seen in Table 3. However, even with these changes in data, each of DMU_1, DMU_2 and DMU_3 maintain their "top ranked" status and hence continue to be DEA fully efficient. Nor is this the end of the line. Continuing with these 5% increments-decrements, while holding w^* fixed, Thompson *et al.* (1994) report that a 15% increment-decrement is needed for a first displacement in which DMU_2 is replaced by DMU_4 and DMU_5. Continuing further, a 20% increment-decrement is needed

Table 3. 5% increments and decrements.

	DMU1	DMU2	DMU3
DMU	$h_j(w^1)$	$h_j(w^2)$	$h_j(w^3)$
1	0.952	0.762	0.381
2	0.680	0.952	0.680
3	0.381	0.762	0.952
4	0.526	0.842	0.702
5	0.702	0.842	0.552
6	0.376	0.526	0.376

to replace DMU_1 with DMU_4 and, finally, still further incrementing and decrementing is needed to replace DMU_3 with DMU_4 as top ranked.

This robust behavior is guaranteed only for a solution which satisfies the "Strong Complementary Slackness Condition" (SCSC) for which a positive gap will appear like ones between the top and second rank shown in every column of Table 2. In fact, the choice of w^* can affect the degree of robustness as reported in Thompson *et al.* (1996) where use of an interior point algorithm produces a w^* closer to the "analytic center" and this considerably increases the degree of robustness for the above example.

Computation Note. Following a recommendation by a referee, we illustrate the computations by using the dual variable values reported in Thompson *et al.* (1994, p. 397) for evaluating DMU_1

$$\begin{aligned}
h_1(w^1) &= \frac{\mu^*}{v_1^* x_{11} + v_2^* x_{12}} = \frac{1}{0.4 + 0.6} = 1.000 \\
h_2(w^1) &= \frac{\mu^*}{v_1^* x_{21} + v_2^* x_{22}} = \frac{1}{0.2 + 1.2} = 0.714 \\
h_3(w^1) &= \frac{\mu^*}{v_1^* x_{31} + v_2^* x_{32}} = \frac{1}{0.1 + 2.4} = 0.400 \\
h_4(w^1) &= \frac{\mu^*}{v_1^* x_{41} + v_2^* x_{42}} = \frac{1}{0.2 + 1.8} = 0.500 \\
h_5(w^1) &= \frac{\mu^*}{v_1^* x_{51} + v_2^* x_{52}} = \frac{1}{0.3 + 1.2} = 0.667 \\
h_6(w^1) &= \frac{\mu^*}{v_1^* x_{61} + v_2^* x_{62}} = \frac{1}{0.4 + 2.4} = 0.357
\end{aligned}$$

These are the values recorded under $h_j(w^1)$ in Table 2. The values in Table 3 are obtained by noting that a 5% increase in input values for DMUs 1, 2, 3 will reduce their $h_j(w^1)$ values to approximately 95%($\approx 1/1.05$) of their previous level. Conversely, a 5% decrement in input values for DMUs 3, 4, 5 will increase their $h_j(w^1)$ values by a factor of 1.05 ($\approx 1/0.95$).

Finally, we should note that all of the dual variable values are positive. See the discussion of the strong complementary slackness condition in the next section of this paper.

8. Strong Complementarity

The "strong complementary slackness condition" (SCSC) plays a central role in the analysis. Hence we briefly recall it in the following form:

Envelopment Slacks / Multiplier Variables		Multiplier Model / Envelopment Variables	
$s_i^{-*} v_i^* = 0$, $s_r^{+*} u_r^* = 0$		$\lambda_j^* t_j^* = 0$	(16.1)
	and		
$s_i^{-*} + v_i^* > 0$, $s_r^{+*} + u_r^* > 0$		$\lambda_j^* + t_j^* > 0$	(16.2)

where $i = 1, \ldots, m; r = 1, \ldots, s;\ j = 1, \ldots, n$.

These variables have all been defined except the t_j^* which represent the slacks associated with the first $j = 1, \ldots, n$ constraints in the multiplier model. The expressions in (16.1) represent necessary conditions for optimum solutions to (10). Hence these conditions will be satisfied by any of the algorithms that are ordinarily used in linear programming. This is not the case for (16.2), however, for which special algorithms like those described in Thompson *et al.* (1996) may be useful and, we might note, Appendix C of Charnes, Cooper and Thrall (1991) provides a procedure for finding whether DMU_j is efficient and, if so, it also provides a finite sequence of optimal basic solutions whose average is SCSC.

Our interest now centers on the expressions on the left in (16). Note that (16.1) requires at least one (and possibly both) of the variables to be zero. Condition (16.2) requires that at least one variable must be positive. It follows that exactly one will be positive. However, here the analyses are based on efficient extreme points. The solutions associated with such points are always unique with all slacks zero in the envelopment model. Hence only the multiplier variables will be positive. It follows that they will all be positive as was the case for the example of DMU_1, above, when SCSC is satisfied. Finally, we might also note that Thompson *et al.* (1994, 1996) refer to this as "full dimensionality" in the resulting solutions where their interest centers on the multiplier variables.

The algorithms used seek to maximize the resulting gap between the first and second DMU. That is, the objective seeks to maximize the value of $d_o(w_o)$ in the following expression,

$$d_o(w_o) = h_o(w_o) - h_k(w_o)$$

where

$$h_k(w_o) \ = \ \max_{j \neq o} h_j(w_o). \tag{17}$$

See the discussion of (14), above.

Thompson *et al.* (1996) refer to this as the case of "DEA Center Solutions." An algorithm needed to ensure the attainment of this DEA Center remains to be developed. To move in this direction they utilize an interior point algorithm (adapted to a DEA context) in order to compute a SCSC solution called the "analytic center" which maximizes the product of the variables. Thus, the fact that it is SCSC maximizes the number of positive variables and the analytic center chooses from this set a solution which maximizes their product. The belief (or hope) is that this will generally be close to the DEA center

Comment. Efficient points which are not extreme can be expressed as convex combinations of efficient extreme points. Hence they will have a zero radius of stability. Such points are of subsidiary interest, however, since they do not enter into the evaluation of other points when the simplex or other extreme point methods of solution are employed (as is the case for most DEA computer codes).

In discussing the algorithms employed by Gonzalez-Lima *et al.* (1996) we might note that necessary as well as sufficient conditions are to be satisfied by the resulting "top rank" characterizations. For further discussion of necessary and sufficient conditions for dual multipliers see Boljuncic and Neralic (1999) who also treat this topic, and supply an algorithm for the case in which the data for one $\text{DMU}_j = \text{DMU}_o$ are worsened (so its efficiency is decreased) and the data for all other DMUs are improved (so their efficiencies are increased). See also Boljuncic (1998).

9. Further Developments

The line of work we now follow returns to envelopment models and in this sense extends the work of Charnes *et al.* (1996) to identify allowable variations in every input and output for every DMU before a change in status occurs for the DMU_o being analyzed. This shift from "multiplier" to "envelopment" models helps to bypass possible concerns which can arise from the different degrees of sensitivity that are associated with alternate optima and different algorithms that might be employed.[5]

The developments for sensitivity analyses we now discuss were initiated by Zhu (1996b) and subsequently extended by Seiford and Zhu (1998b). However, we start our discussion of this path of development (which revolves around the uses of envelopment models) with the later models due to Seiford and Zhu (1998b) because the earlier paper by Zhu (1996b) was shown to be vulnerable to counter example by Boljuncic (1999). Hence we start with the following version of the CCR model as formulated in Seiford and Zhu (1998b),

$$\beta^* = \min \beta$$

subject to

$$\sum_{j=1, j\neq 0}^{n} x_{ij}\lambda_j \leq \beta x_{io}, \qquad i \in I$$

$$\sum_{j=1, j\neq 0}^{n} x_{ij}\lambda_j \leq x_{io}, \qquad i \notin I$$

$$\sum_{j=1, j\neq 0}^{n} y_{rj}\lambda_j \geq y_{ro}, \qquad r = 1, \ldots, s$$
$$\beta, \lambda_j \geq 0. \tag{18}$$

Here the set $i \in I$ consists of inputs where sensitivity is to be examined and $i \notin I$ represents inputs where sensitivity is not of interest.

Seiford and Zhu use this model to determine ranges of data variation when inputs are worsened for DMU_o in each of its x_{io} and improved for the x_{ij} of every DMU_j, $j = 1, \ldots, n$ in the set $i \in I$. We sketch the development by introducing the following formulation to determine the range of admissible variations,

$$\sum_{j=1, j\neq o}^{n} \frac{x_{ij}}{\delta}\lambda_j \leq \beta\delta x_{io}, \qquad i \in I$$

where

$$1 \leq \delta \leq \beta^* \tag{19}$$

Now assume that we want to alter these data to new values $\hat{x}_{io} \geq x_{io}$ and $\hat{x}_{ij} \leq x_{ij}$. To examine this case we use

$$\sum_{j=1, j\neq o}^{n} \hat{x}_{ij}\lambda_j \leq \beta\hat{x}_{io}$$

where

$$\begin{aligned} \hat{x}_{io} &= x_{io} + \delta x_{io} - x_{io} = x_{io} + (\delta - 1)x_{io} \\ \hat{x}_{ij} &= \frac{x_{ij}}{\delta} = x_{ij} + \frac{x_{ij}}{\delta} - x_{ij} = x_{ij} - \left(\frac{\delta - 1}{\delta}\right)x_{ij} \end{aligned} \tag{20}$$

for every $j = 1, \ldots, n$ in the set $i \in I$.

Thus $(\delta - 1)$ represents the proportional *increase* to be allowed in each x_{io} and $(\delta - 1)/\delta$ represents the proportional *decrease* in each x_{ij}, $j \neq o$. As proved by Seiford and Zhu, the range of variation that can be allowed for δ without altering the efficient status of DMU_o is given in the following

THEOREM 5 (Seiford and Zhu) *If* $1 \leq \delta \leq \sqrt{\beta^*}$ *then* DMU_o *will remain efficient. That is, any value of* δ *within this range of proportional variation for both the* x_{io} *and* x_{ij} *will not affect the efficient status of* DMU_o.

Here δ is a parameter with a value to be selected by the user. The theorem asserts that no choice of δ within the indicated range will cause DMU_o to be reclassified as inefficient when the $\hat{x}_{io}$ and $\hat{x}_{ij}$ defined in (20) are substituted in (18) because the result will still give $\beta^* \geq 1$.

The lower limit of $\delta \geq 1$ is needed to ensure that consideration is being given to both input worsenings for DMU_o and input improvements for the DMU_j, $j \neq o$. If the upper limit is breached a value of $\beta^* < 1$ will be achieved when substutions are effected from (20) into (18). DMU$_o$ will then be moved from efficient to inefficient status.

Seiford and Zhu supply a similar development for outputs and then join the two in the following model which permits simultaneous variations in inputs and outputs,

$$
\begin{aligned}
&\gamma^* = \min \gamma \\
&\text{subject to} \\
&\sum_{j=1, j\neq 0}^{n} x_{ij}\lambda_j \le (1+\gamma)x_{io}, \qquad i \in I \\
&\sum_{j=1, j\neq 0}^{n} x_{ij}\lambda_j \le x_{io}, \qquad i \notin I \\
&\sum_{j=1, j\neq 0}^{n} y_{rj}\lambda_j \ge (1-\gamma)y_{ro}, \qquad r \in S \\
&\sum_{j=1, j\neq 0}^{n} y_{rj}\lambda_j \ge y_{ro}, \qquad r \notin S \\
&\lambda_j \ge 0, \qquad j = 1, \ldots, n; \quad j \neq 0; \quad \gamma \text{ unrestricted.}
\end{aligned} \tag{21}
$$

where $i \in I$ represents the input set for which data variations are to be considered and $r \in S$ represents the output set for which data variations are to be considered. Using δ to represent allowable input variations and τ to represent allowable output variations, Seiford and Zhu supply the following

THEOREM 6 (Seiford and Zhu) *If* $1 \le \delta \le \sqrt{1+\gamma^*}$ *and* $\sqrt{1-\gamma^*} \le \tau \le 1$ *then* DMU$_o$ *will remain efficient.*

As Seiford and Zhu (1998b) note, for $I = \{1, \ldots, m\}$ and $S = \{1, \ldots, s\}$ (21) is the same as the CCR correspond used in Charnes *et al.* (1992, 1996). Seiford and Zhu have thus generalized these results from Charnes *et al.* to allow simultaneous variations in all inputs and outputs for every DMU in the sets $i \in I$ and $r \in S$ in a manner that now provides an alternative to the earlier work by Thompson *et al.* (1994, 1996) which we discussed in the preceding section. We do not further discuss (21) and the same is true for the following model—which Seiford and Zhu use to treat the case where absolute (rather than proportional) changes in the data are of interest,

$$
\begin{aligned}
&u^* = \min u \\
&\text{subject to} \\
&\sum_{j=1, j\neq 0}^{n} x_{ij}\lambda_j \le x_{io} + u, \qquad i \in I \\
&\sum_{j=1, j\neq 0}^{n} x_{ij}\lambda_j \le x_{io}, \qquad i \notin I
\end{aligned}
$$

Table 4. Comparison of Seiford and Zhu with Thompson-Thrall *et al.*

	DMU1	DMU2	DMU3
(g_o, g)	(41, 29)	(12, 11)	(41, 29)
SCSC1	20	14	20
SCSC2	32	9	32

Source: Seiford and Zhu (1998).
Here $g_o = \delta - 1$ and $g = \frac{\delta-1}{\delta}$. See (20).

$$\sum_{j=1, j\neq 0}^{n} y_{rj}\lambda_j \geq y_{ro} - u, \qquad r \in s$$
$$\sum_{j=1, j\neq 0}^{n} y_{rj}\lambda_j \geq y_{ro}, \qquad r \notin s$$
$$\sum_{j=1, j\neq 0}^{n} \lambda_j = 1$$
$$u, \lambda_j \geq 0, \quad \forall j. \tag{22}$$

This can be regarded as an extension of (6) in which variations are to be undertaken only for subsets of the data as in, for instance, the constraints with non-zero slacks and with the thus identified subset varying for different DMUs.

We now turn to Table 4 which Seiford and Zhu use to compare their approach with the Thompson *et al.* (1996) approach. To interpret this Table we note that all results represent percentages in the allowed data variations by applying these two different approaches to the data of Table 1 given in section 7, above. The values in the rows labeled SCSC1 and SCSC2 are secured from two alternate optima which Thompson *et al.* (1996) report as satisfying the strong complementary slackness condition when only the inputs listed in Table 1 are varied. The parenthesized values of g_o and g at the top of Table 4 are the percentages reported by Seiford and Zhu as having been obtained by applying (18) to these same data

Examples. Using the data from Table 1 in section 7 we omit DMU_1 from the right hand side in order to generate the following model from (18) to determine what Charnes, Rousseau and Semple (1996) refer to as a "Radius of Classification Preservation" (RCP) for DMU_1,

$$\beta^* = \min \beta$$

subject to

$$4\beta = 2\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 + 4\lambda_6 + s_1^-$$
$$1\beta = 2\lambda_2 + 4\lambda_3 + 3\lambda_4 + 2\lambda_5 + 4\lambda_6 + s_2^-$$
$$1 = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - s^+. \tag{A}$$

With all variables constrained to be non-negative this has $\lambda_2^* = 1$, $\hat{s}_1^{-*} = 6$, $\beta^* = 2$ as a solution.

A proportional increase of 100% or more is evidently required to change DMU_1 from efficient to inefficient status as evidenced by $\beta^* = 2$. To confirm this we replace (A) with the following formulation

$$
\begin{aligned}
&\theta^* = \min \theta \\
&\text{subject to} \\
&8\theta = 2\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 + 4\lambda_6 + s_1^- \\
&2\theta = 2\lambda_2 + 4\lambda_3 + 3\lambda_4 + 2\lambda_5 + 4\lambda_6 + s_2^- \\
&1 = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - s^+ \\
&0 \le \lambda_2, \ldots, \lambda_6, s_1^-, s_2^-, s^+.
\end{aligned}
\tag{B}
$$

This has a solution with $\theta^* = 1$, $\lambda_2^* = 1$, $s_1^{-*} = 6$. In the terminology of Andersen and Petersen (1993), DMU_1 loses its "super-efficiency" status. With these input values it is possible for the other DMUs to produce its one unit of output without any input augmentation.

Following Seiford-Zhu we now choose $\delta^2 = \beta^* = 2$ in (A). That is, we are choosing the upper limit allowed by theorem 5 of Seiford and Zhu, as given after (19) and (20) and then dividing through by $\delta = \sqrt{\beta^*} \approx 1.41$ to replace (B) with

$$
\begin{aligned}
&\theta^* = \min \theta \\
&\text{subject to} \\
&5.64\theta = 1.41\lambda_2 + 0.71\lambda_3 + 1.41\lambda_4 + 2.13\lambda_5 + 2.84\lambda_6 + s_1^- \\
&1.41\theta = 1.41\lambda_2 + 2.84\lambda_3 + 2.13\lambda_4 + 1.41\lambda_5 + 2.84\lambda_6 + s_2^- \\
&1 = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - s^+.
\end{aligned}
\tag{C}
$$

which has a solution with $\lambda_2^* = 1$, $s_1^{-*} = 4.23$, $\theta^* = 1$ and all other variables zero.[6]

The 100% RCP (= Radius of Classification Preservation) for DMU_1 in (A) is replaced by only a 41% augmentation of the inputs for DMU_1 in (C). The latter value obtained from the Seiford-Zhu approach is, however, accompanied by a 29% $(= 1 - 1.41/2) \times 100\%$ improvement in the inputs of DMU_2. Both adjustments are adverse to DMU_1 but its sensitivity is nevertheless increased in comparison to the stationary frontier used in the Charnes *et al.* approaches.

The results from Seiford and Zhu also seem to be more robust than is the case for Thompson, Thrall and their associates—see Thompson *et al.* (1994, 1996)—at least for DMU_1 and DMU_3. This is not true for DMU_2, however, where a 14% worsening of its inputs and 14% improvement in the inputs of the non-efficient DMUs is required under the Thompson *et al.* (1996) approach before DMU_2 will change from efficient to inefficient in its status. However, the Seiford and Zhu approach shows that DMU_2 will retain its efficient status until at least a 12% worsening of its 2 inputs occurs along with an 11% improvement in these same inputs for the inefficient DMUs. A range of $12\% + 11\% = 23\%$ does not seem to be far out of line with the $2 \times 14\% = 28\%$ or the $2 \times 9\% = 18\%$ reported by Thompson *et al.* (1996). Moreover, as Seiford and Zhu note, their test is more severe. They match their worsening of DMU_2's inputs with improvement of the inputs of *all* of the other DMUs—including the efficient DMU_1 and DMU_2—whereas Thompson, Thrall, *et al.* worsen the inputs of *all* of the efficient DMUs and improve the inputs of only the inefficient DMUs.[7]

Comment. In a further extension of Seiford and Zhu (1998b), it is shown by Zhu (1999) that data perturbations associated with δ can be decomposed into components. As a result, DEA sensitivity analysis can be done (i) in a general situation where data for a test DMU and data for the remaining DMUs are allowed to vary simultaneously and unequally and (ii) in a worst-case scenario where the efficiency of the test DMU is deteriorating while the efficiencies of the other DMUs are improving.

There are many more developments in these approaches by Zhu and Seiford, Charnes *et al.* and Thompson *et al.* but they cannot be covered here. We do need to note, however, that Seiford and Zhu extend their results to deal with the infeasibility that can occur when the sums to be considered omit the $j = o$ being evaluated.[8] They show that infeasibility means that the DMU_o being tested will preserve its efficient status in the presence of infinite increases in its inputs and infinite decreases in its outputs.[9]

10. Summary and Conclusion

Thompson *et al.* (1994) found results from the DEA efficiency analysis to be robust with respect to efficient and inefficient DMUs when DEA was applied to data on Kansas farms and Illinois coal mines. This same result was obtained in the Thompson *et al.* (1996) study of independent oil companies. Zhu (1996a) similarly reported robust results in his study of a Chinese textile company as did Seiford and Zhu (1998b, 1998c) in use of data on the efficiencies of Chinese cities.[10] Hopefully, continuing work along these lines will help point the way toward substantive generalizations about the robustness properties (or lack thereof) in DEA.

As we have already observed, the progress in the sensitivity analysis studies we discussed has effected improvements in two important directions. First, this work has moved from evaluating one input or one output at a time in one DMU and has proceeded into more general situations where all inputs and outputs for all DMUs can be simultaneously varied. Second, the need for special algorithms and procedures (other than those already incorporated in DEA computer codes) has been reduced or eliminated at least in the formulations by Seiford and Zhu but not in the approaches of Thompson *et al.* See also the references in Neralic (1997) for references to the algorithmic developments which he has undertaken. See also Boljuncic (1998).

Altering the focus to points which are weakly efficient, as in Seiford and Zhu, assumes that non-zero slacks are of no consequence. This avoids the problem of alternate optima but raises other problems in its place. One could, of course, move to additive models with their associated ℓ_1 metrics as is done in Charnes *et al.* to treat such non-zero slacks. However, as discussed in Thrall (1996), this approach brings with it the problem of a possible lack of invariance of solutions when the units used to measure inputs or outputs are changed.

There is also a need to extend the Seiford and Zhu analysis (or something like it) to provide measures of stability for inefficient as well as efficient DMUs. A sharpening of results would also be welcome. For instance, unlike the approaches which we have discussed in Thompson *et al.* (1994, 1996), the models and methods used by Seiford and Zhu identify only the conditions under which DMU_o loses its efficiency status. That is, unlike Thompson *et al.*, Seiford and Zhu do not identify which DMUs effect the indicated displacements.

The work discussed in this paper deals only with changes of classification from efficient to inefficient status (or *vice versa*). Extensions might well be effected that move from these qualitative characterization in order to determine the differing amounts of inefficiencies for the inefficient DMUs affected by such changes in classification. (A good start in this direction is made in Thompson *et al.* (1996) but more is needed.)

By and large, these analytical approaches to sensitivity have restricted attention to issues of technical efficiency. A start toward other kinds of analyses is to be found in the paper by Seiford and Zhu (1999) which analyses the sensitivity of returns to scale characterizations in DEA. Extensions of sensitivity analyses to other types of efficiency (such as allocative efficiency and congestion) are yet to be made.

Another topic of interest revolves around effects that might be associated with deleting or adding DMUs. Wilson (1995) utilizes the approaches of Andersen-Petersen (1993) to study the effects of removing DMUs in order to determine "influential observations" (= DMUs). The effects of adding DMUs might also be studied with respect to their effects on efficiency scores. (See Thrall (1989) for a good start on this topic.) Also of interest would be the effects of simultaneously adding and deleting DMUs—such as occur with the "window analyses" discussed in Chapter IX of Cooper, Seiford and Tone (1999).[11] This could also be extended in yet another manner as is in Cherchye, Kuosmanen and Post (2000) who assign probability measures to the maximum number of DMU_o that can be removed from a data set without altering the efficiency status of any specified DMU_o. In yet another direction, Charnes, Cooper and Rhodes (1980) employ general gamma distributions to study the statistical behavior of efficiency scores—which include effects that accompany the projections of original observations onto efficiency frontiers when it is desired to distinguish between the efficiency of programs and the efficiency with which the programs were managed.

We might also turn to questions of sample size, or number of observations, which we discussed in our introduction. One would like theorems with accompanying "substantive" characterizations of stability like those reported in Banker (1993) for the statistical consistency of DEA results with varying sample size or, alternatively, one could proceed with bootstrap methods like those discussed in Simar and Wilson (2000).

More, and more varied, applications of these ideas could help to establish whether DEA results are empirically robust and, perhaps, identify conditions where the results are likely to be sensitive. See Raab, Kotamrajv and Haage (2000) for a recent addition to the uses of these ideas we referenced earlier in empirical work reported in Thompson *et al.* (1994, 1996) and Zhu (1996b). Improved algorithms along lines like those we covered in our discussion of the Thompson *et al.* (1994, 1996) methods and the Gonzales-Lima *et al.* (1996) studies and, of course, exploitation of algorithms like those reported or referenced in Neralic (1997) could help to expand the number and variety of such applications, especially if this work were incorporated in some of the DEA computer codes that are now being extensively used in a wide variety of applications.

Comment. There are also technical problems of possible interest which include relations of duality that remain to be explored. One possibility along these lines was called to our attention by a referee who noted that the multiplier values sum to ≤ 1 for efficient DMUs in the dual to (5) and they sum to ≥ 1 for inefficient DMUs in the dual to (2). Reference to (1) suggests that this can be regarded as an extension of the condition that the input (or output) multipliers sum to 1 when applied to the inputs or outputs as in the last expression on the right in (10) but it brings with it added properties that remain to be explored.

There are, of course, numerous other possibilities for further progress and new problems to be dealt with. Some of this further progress is evidenced by other papers that were presented at the workshop in Copenhagen. See Boljuncic and Neralic (1999). Progress is also possible on topics that we have not covered. For instance, we have not covered topics like sensitivity to changes in the models used[12] or the input and output variables used.[13] It is not possible to cover all of such work in a single paper, but additional approaches to other problems in sensitivity analysis may be found in the references. See, e.g., the discussion of "envelopment maps" for use in sensitivity analysis that is described in Bulla *et al.* (2000) as well as the treatment for the effects of adding variables in Thrall (1989).

Acknowledgement

This is a revised version of a paper presented to the Sixth European Workshop on Efficiency and Productivity Analysis at the Royal Agricultural University in Copenhagen, Denmark, October 29–31, 1999. Comments by the editor, C. A. Knox Lovell, and an anonymous referee led to numerous improvements. Support from the IC2 Institute of the University of Texas is gratefully acknowledged. Acknowledgement is also due to Kluwer Academic Publishers for permission to reproduce material from *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software* by W. W. Cooper, L. Seiford and K. Tone (Norwell, MA: Kluwer Academic Publishers, 1999).

Notes

1. As cited in Charnes *et al.* (1992a), use of this concept is adapted from Zlobec, Gardner and Ben Israel (1981).
2. See Cooper *et al.* (2000) for a use of such boundaries to obtain evaluations of the efficiency with which congestion is managed in Chinese production in situations where such congestion is a result of governmental policies directed to ensuring employment of a huge and growing labor force.
3. This omission of DMU_o is also used in Andersen and Petersen (1993) as discussed in the next section.
4. The ranges within which these dual variable values do not change form part of the printouts in standard linear programming computer codes.

5. An additional alternative is to combine the two approaches (i.e., envelopment model and multiplier model approaches) as discussed in Boljuncic and Neralic (1999).
6. $\lambda_5^* = 1, s_1^{-*} = 3.51$ is also optimal with $\theta^* = 1$.
7. The fact that Seiford and Zhu deal only with "weak efficiency" is not pertinent here because DMU_1, DMU_2 and DMU_3 are all strongly efficient.
8. Seiford and Zhu (1998a) note that the possibility of infeasibility is not confined to the case when convexity is imposed. It can also occur when certain patterns of zeros are present in the data.
9. Refinements are evidently needed to exclude the possibility of negative outputs but we do not treat this topic here.
10. These data were taken from Charnes, Cooper and Li (1989).
11. See also the discussion of "envelopment maps" which, as discussed in Cooper, Seiford and Tone (1999), can provide a start toward such analyses.
12. For an example see Ahn and Seiford (1993).
13. See the study by R. D. Banker, H. Chang and W. W. Cooper (1996) which deals with the effects of misspecified variables in DEA.

References

Ahn, T., A. Charnes, and W. W. Cooper. (1988). "A Note on the Efficiency Characterizations Obtained in Different DEA Models." *Socio-Economic Planning Sciences* 6, 253–257.

Ahn, T., and L. M. Seiford. (1993). "Sensitivity of DEA to Models and Variable Sets in an Hypothesis Test Setting: The Efficiency of University Operations." In Y. Ijiri (ed.), *Creative and Innovative Approaches to the Science of Management*. New York: Quorum Books.

Andersen, P., and N. C. Petersen. (1993). "A Procedure for Ranking Efficient Units in Data Envelopment Analysis." *Management Science* 39, 1261–1264.

Banker, R. D. (1993). "Maximum Likelihood, Consistency and Data Envelopment Analysis: A Statistical Foundation." *Management Science* 39, 1265–1273.

Banker, R. D., H. Chang, and W. W. Cooper. (1996). "Simulation Studies of Efficiency, Returns to Scale and Misspecification with Nonlinear Functions in DEA." *Annals of Operations Research* 66, 233–253.

Banker, R. D., A. Charnes, W. W. Cooper, and A. Maindiratta. (1988). "A Comparison of DEA and Translog Estimates of Production Frontiers Using Simulated Observations from a Known Technology." In A. Dogramaci and R. Färe (eds.), *Applications of Modern Production Theory: Efficiency and Productivity*. Boston: Kluwer Academic Publishers.

Boljuncic, V. (1999). "A Note on Robustness of the Efficient DMUs in Data Envelopment Analysis." *European Journal of Operational Research* 112, 240–244.

Boljuncic, V. (1998). "Sensitivity Analysis in the Additive Model of Data Envelopment Analysis." *International Journal of System Sciences* 20, 219–222.

Boljuncic, V., and L. Neralic. (1999). "Evaluation of Robustness of Decision Making Units Using Dual Multipliers." Paper presented at 6th European Workshop on Productivity and Efficiency, Royal Agricultural University, Copenhagen, Denmark, 10/29–10/31/99.

Bulla, S. P., W. W. Cooper, K. S. Park, and D. Wilson. (2000). "Evaluating Efficiencies of Turbofan Jet Engines in Multiple Input-Output Contexts: A Data Envelopment Analysis Approach." *Journal of Propulsion and Power* 6 (to appear).

Charnes, A., and W. W. Cooper. (1961). *Management Models and Industrial Applications of Linear Programming*. New York: John Wiley and Sons, Inc.

Charnes, A., and W. W. Cooper. (1968). "Structural Sensitivity Analysis in Linear Programming and an Exact Product Form Left Inverse." *Naval Research Logistics Quarterly* 15, 517–522.

Charnes, A., W. W. Cooper, D. Divine, T. W. Ruefli, and D. Thomas. (1989). "Comparisons of DEA and Existing Ratio and Regression Systems for Effecting Efficiency Evaluations of Regulated Electric Cooperation in Texas." *Research in Governmental and Nonprofit Accounting* 5, 187–210.

Charnes, A., W. W. Cooper, B. Golany, L. Seiford, and J. Stutz. (1985). "Foundations of Data Envelopment Analysis for Pareto-Koopmans Efficient Empirical Production Functions." *Journal of Econometrics* 30, 91–107.

Charnes, A., W. W. Cooper, A. Y. Lewin, R. C. Morey, and J. J. Rousseau. (1985). "Sensitivity and Stability Analysis in DEA." *Annals of Operations Research* 2, 139–150.

Charnes, A., W. W. Cooper, and S. Li. (1989). "Using DEA to Evaluate Relative Efficiencies in the Economic Performance of Chinese Cities." *Socio-Economic Planning Sciences* 23, 325–344.

Charnes, A., W. W. Cooper, and E. Rhodes. (1978). "Measuring the Efficiency of Decision Making Units." *European Journal of Operational Research* 2, 429–444.

Charnes, A., W. W. Cooper, and E. Rhodes. (1980). Appendix: "The Distribution of DMU Efficiency Measures." In A. Charnes and W. W. Cooper. (1980). "Management Science Relations for Evaluation and Management Accountability." *Journal of Enterprise Management* 2, 160–162.

Charnes, A., W. W. Cooper, and R. M. Thrall. (1991). "A Structure for Characterizing and Classifying Efficiencies in DEA." *Journal of Productivity Analysis* 3, 197–237.

Charnes, A., W. W. Cooper, and R. M. Thrall. (1986). "Identifying and Classifying Scale and Technical Inefficiencies in Observed Data and Data Envelopment Analysis." *Operations Research Letter* 5, 105–110.

Charnes, A., S. Haag, P. Jaska, and J. Semple. (1992a). "Sensitivity of Efficiency Calculations in the Additive Model of Data Envelopment Analysis." *Journal of Systems Sciences* 23, 789–798.

Charnes, A., and L. Neralic. (1992b). "Sensitivity Analysis of the Proportionate Change of Inputs (or Outputs) in Data Envelopment Analysis." *Glasnik Matematicki* 27, 393–405.

Charnes, A., J. J. Rousseau, and J. H. Semple. (1996). "Sensitivity and Stability of Efficiency Classifications in DEA." *Journal of Productivity Analysis* 7, 5–18.

Cherchye, L., T. Kuosmanen, and T. Post. (2000). "New Tools for Dealing with Errors-in-Variables in DEA." Discussion Paper Series DPS 00.06. Leuven, Belgium: Katholieke Universiteit Leuven, Dept. of Economics.

Cooper, W. W., H. Deng, B. Gu, and S. Li. (2000). "Using DEA to Improve the Management of Congestion in Chinese Industries (1981–1997)." *Socio-Economic Planning Sciences* (to appear 2001).

Cooper, W. W., K. S. Park, and J. Pastor. (1999). "RAM: A Range Adjusted Measure of Inefficiency for Use with Additive Models and Relations to Other Models and Measures in DEA." *Journal of Productivity Analysis* 11, 5–42.

Cooper, W. W., L. M. Seiford, and K. Tone. (1999). *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*. Norwell, Mass.: Kluwer Academic Publisher.

Dula, J. H., and B. L. Hickman. (1997). "Effects of Excluding the Column Being Scored from the DEA Envelopment Technology Matrix." *Journal of the Operational Research Society* 48, 1001–1012.

Gonzales-Lima, M. D., R. A. Tapia, and R. M. Thrall. (1996). "On the Construction of Strong Complementarity Slackness Solutions for DEA Linear Programming Problems Using a Primal-Dual Interior-Point Method." *Annals of Operations Research* 66, 139–162.

Grosskopf, S. (1996). "Statistical Inference and Nonparametric Efficiency: A Selective Survey." *Journal of Productivity Analysis*, 161–176.

Joro, T., P. Korhonen, and J. Wallenius. (1998). "Structural Comparison of Data Envelopment Analysis and Multiple Objective Programming." *Management Science* 44, 962–970.

Neralic, L. (1997). "Sensitivity in Data Envelopment Analysis for Arbitrary Perturbations of Data." *Glasnik Matematicki* 32, 315–335.

Neralic, L. (2000). "Preservation of Efficiency and Inefficiency Classification in Data Envelopment Analysis." *Research Report* Faculty of Economics, University of Zagreb, Kennedyev TRG 6, 1000 Zagreb, Croatia.

Post, T. (1999). *Finding the Frontier: Methodological Advances in Data Envelopment Analysis*. Rotterdam: Tinbergen Institute Research Series. Erasmus University.

Raab, R., D. Kotamrajv, and S. Haag. (2000). "Efficient Provision of Child Quality of Life in Less Developed Contries. Conventional Development Indexes vs. A Programming Approach to Development Indexes." *Socio-Economic Planning Sciences* 34, 51–67.

Seiford, L. M. (1994). "A DEA Bibliography (1978–1992)." In A. Charnes, W. W. Cooper, A. Lewin, and L. M. Seiford (eds.), Chapter 22 in *Data Envelopment Analysis: Theory, Methodology and Applications*. Norwell, Mass: Kluwer Academic Publishers.

Seiford, L. M., and J. Zhu. (1998a). "Infeasibility of Super-Efficiency Data Envelopment Analysis Models." *INFOR* 37, 174–187.

Seiford, L. M., and J. Zhu. (1998b). "Sensitivity Analysis of DEA Models for Simultaneous Changes in All of the Data." *Journal of the Operational Research Society* 49, 1060–1071.

Seiford, L. M., and J. Zhu. (1999). "Sensitivity Analysis of Returns to Scale Classifications in DEA." *Journal of Productivity Analysis* 12, 55–75.

Seiford, L. M., and J. Zhu. (1998c). "Stability Regions for Maintaining Efficiency in Data Envelopment Analysis." *European Journal of Operational Research* 108, 127–139.

Sexton, T. R., R. H. Silkman, and R. H. Hogan. (1986). "Measuring Efficiency: An Assessment of Data Envelopment Analysis, New Directions for Program Evaluations." In R. H. Silkman (ed.), *Measuring Efficiency: An Assessment of Data Envelopment Analysis*. Publication No. 32 in the series New Directions for Program Evaluations. A publication of the American Evaluation Association. San Francisco: Jossy Bass.

Simar, L., and P. W. Wilson. (1998). "Sensitivity Analysis of Efficiency Scores: How to Bootstrap in Nonparametric Frontier Models." *Management Science* 44, 49–61.

Simar, L., and P. W. Wilson. (2000). "Statistical Inference in Nonparametric Frontier Models: The State of the Art." *Journal of Productivity Analysis* 13, 49–78.

Thompson, R. G., P. S. Dharmapala, J. Diaz, M. D. Gonzalez-Lima, and R. M. Thrall. (1996). "DEA Multiplier Analytic Center Sensitivity Analysis with an Illustrative Application to Independent Oil Cos." *Annals of Operations Research* 66, 163–180.

Thompson, R. G., P. S. Dharmapala, and R. M. Thrall. (1994). "Sensitivity Analysis of Efficiency Measures with Applications to Kansas Farming and Illinois Coal Mining." In A. Charnes, W. W. Cooper, A. Y. Lewin, and L. M. Seiford (eds.), *Data Envelopment Analysis: Theory, Methodology and Applications*. Norwell, Mass.: Kluwer Academic Publishers, pp. 393–422.

Thrall, R. M. (1989). "Classification of Transitions Under Expansion of Inputs and Outputs." *Managerial and Decision Economics* 10, 159–162.

Thrall, R. M. (1996). "Duality, Classification and Slacks in DEA." *Annals of Operations Research* 66, 104–138.

Thrall, R. M. (2000). "Measures in DEA with an Application to the Malmquist Index." *Journal of Productivity Analysis* 13, 125–138.

Wilson, P. W. (1995). "Detecting Influential Observations in Data Envelopment Analysis." *Journal of Productivity Analysis* 6, 27–46.

Zhu, J. (1996a). "DEA/AR Analysis of the 1988–1989 Performance of Nanjing Textile Corporation." *Annals of Operations Research* 66, 311–335.

Zhu, J. (1996b). "Robustness of the Efficient DMUs in Data Envelopment Analysis." *European Journal of Operational Research* 90, 451–460.

Zhu, J. (1999). "Super Efficiency and DEA Sensitivity." Working Paper, Department of Management, Worcester Polytechnic Institute, Worcester, Mass. 01609.

Zlobec, S., R. Gardner, and A. Ben-Israel. (1981). "Regions of Stability for Arbitrarily Perturbed Convex programs." In A. Fiacco (ed.), *Mathematical Programming with Data Perturbations*. New York: Marcel Dekker, pp. 69–89.