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An investigation of returns to scale in data envelopment analysis

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Abstract

This paper discusses the determination of returns to scale (RTS) in data envelopment analysis (DEA). Three basic RTS methods and their modifications are reviewed and the equivalence between these different RTS methods is presented. The effect of multiple optimal DEA solutions on the RTS estimation is studied. It is shown that possible alternate optimal solutions only affect the estimation of RTS on DMUs which should be classified as constant returns to scale (CRS). Modifications to the original RTS methods are developed to avoid the effects of multiple optimal DEA solutions on the RTS estimation. The advantages and disadvantages of these alternative RTS methods are presented so that a proper RTS method can be selected within the context of different applications.
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1. Introduction

It has long been recognized that data envelopment analysis (DEA) by its use of mathematical programming is particularly adept at estimating multiple input and multiple output production correspondences. Since the first DEA model, the CCR model [1], a number of different DEA models have appeared in the literature [2]. During this period of model development, the economic notion of returns-to-scale (RTS) was widely studied within the framework of DEA. This, in turn, further extended the applicability of DEA. For example, Banker *et al.* [3] studied the 114 North Carolina hospitals' production functions and efficiencies and uncovered the possibilities of RTS on individual hospitals while the previous (regression-based) study had reached the conclusion that no RTS were present.

As a result of this research thrust, there are at least three different *basic* methods of testing a decision making unit (DMU)'s RTS nature which have appeared in the DEA literature. Banker [4] shows that the CCR model can be employed to test for DMUs' RTS using the concept of most productive scale size (MPSS), i.e. the sum of the CCR optimal lambda values can determine the RTS classification. We call this method the *CCR RTS method*. Banker *et al.* (BCC) [5] report that a new free BCC dual variable (u_o) estimates RTS by allowing variable returns to scale (VRS) for the CCR model, i.e. the sign of u_o determines the RTS. We call this method the *BCC RTS method*. Finally, Färe *et al.* [6] provide the *scale efficiency index method* for the determination of RTS using DEA. These three RTS methods, in fact, are equivalent but different presentations (see, for example, Refs. [7–9]).

The three basic RTS methods have been widely employed in real world situations (see, for example, Refs. [10–12]). However, it has been noted that the CCR and BCC RTS methods may fail when DEA models have alternate optima, i.e. the original CCR

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and BCC RTS methods assume unique optimal solutions to the DEA formulations. In contrast to the CCR and BCC RTS methods, the scale efficiency index method does not require information on the primal and dual variables and, in particular, is robust even when there exist multiple optima. Since it may be impossible or at least unreasonable to generate all possible multiple optima in many real world applications, a number of modifications or extensions of the original CCR and BCC methods have been developed to deal with multiple optima.

Banker and Thrall [13] generalize the BCC RTS method by exploring all alternate optima in the BCC dual model, i.e. RTS in their extended technique is measured by intervals for u_o. Banker *et al.* [14] further modified the technique to avoid the need for examining all alternate optima. Using the same technique, Banker *et al.* [7] introduce a modification to the CCR RTS method by determining the maximum and minimum values of $\Sigma_{j=1}^{n}\lambda_j$ in the CCR model in order to reach a decision. On the other hand, by the scale efficiency index method, Zhu and Shen [15] suggest a remedy for the CCR RTS method under possible multiple optima.

In addition to reviewing the existing RTS methods, the current paper provides some computationally simple methods to characterize RTS and to circumvent the need for exploring all alternate optimal solutions to the CCR primal model and to the BCC dual model. It can be seen that on the basis of these newly modified CCR and BCC RTS methods, one avoids the possible effect of multiple optimal DEA solutions on the determination of RTS. One also avoids the need for solving the DEA formulations for determining the maximum and minimum values of $\Sigma_{j=1}^{n}\lambda_j$ as described in Ref. [7] and of u_o as described in Ref. [13].

The paper is organized as follows. Section 2 provides the DEA models that are employed in the three basic RTS methods. Then in Section 3 we discuss the definitions of RTS in DEA. This is followed by a review of the three basic RTS methods and their equivalence. We address the issue of multiple optima in the CCR primal and BCC dual formulations. It is shown that possible alternate optimal solutions only affect the estimation of RTS on DMUs which should be classified as constant returns to scale (CRS). Finally, we present and develop the modifications to the original CCR and BCC RTS methods. We also discuss the implementation of different modifications. Conclusions and possible further research are given in Section 6.

2. DEA models

In order to develop our discussion of RTS, we present the related (input-oriented) DEA models that are required for the three basic RTS methods (see Ref. [16] for a detailed discussion of these DEA models). Suppose we have n DMUs. Each DMU$_j$, $j = 1, 2, \ldots, n$ produces s different outputs, y_{rj} ($r = 1, 2, \ldots, s$), using m different inputs, x_{ij} ($i = 1, 2, \ldots, m$). Then the primal linear program for the (input-based) CCR model can be written as

$$\begin{aligned} &\theta^* = \min \theta \\ \text{s.t.}\quad &\sum_{j=1}^{n}\lambda_j x_{ij} \le \theta x_{io} \quad i = 1, 2, \ldots, m; \\ &\sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{ro} \quad r = 1, 2, \ldots, s; \\ &\lambda_j \ge 0 \quad j = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

where x_{io} and y_{ro} are, respectively, the ith input and rth output for DMU$_o$ under evaluation. Associated with the $m + s$ input and output constraints, some non-zero slacks may be identified by utilizing a two-stage process where the efficiency score is first calculated and then the sum of slacks is maximized [17].

The dual linear program to Eq. (1) is

$$\begin{aligned} \max\quad &\sum_{r=1}^{s}u_r y_{ro} \\ \text{s.t.}\quad &\sum_{r=1}^{s}u_r y_{rj} - \sum_{i=1}^{m}v_i x_{ij} \le 0 \quad j = 1, \ldots, n; \\ &\sum_{i=1}^{m}v_i x_{io} = 1 \\ &u_r,\ v_i \ge 0 \end{aligned} \tag{2}$$

A DMU$_o$ is said to be CCR efficient *if and only if* (a) $\theta^* = 1$ and (b) all optimum slack values in Eq. (1) are zero.

Associated with the CCR model, we have

Proposition 1. *At least one DMU is CCR efficient.*

Another DEA model, which is usually referred to as the BCC model, can be expressed as

$$\begin{aligned} &b^* = \min b \\ \text{s.t.}\quad &\sum_{j=1}^{n}\lambda_j x_{ij} \le b x_{io} \quad i = 1, 2, \ldots, m; \\ &\sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{ro} \quad r = 1, 2, \ldots, s; \\ &\sum_{j=1}^{n}\lambda_j = 1 \\ &\lambda_j \ge 0 \quad j = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

The dual to the above linear program is

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r y_{ro} + u_o \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0 \quad j = 1, \ldots, n; \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & u_r,\ v_i \geq 0 \text{ and } u_o \text{ is free} \end{aligned} \tag{4}$$

A DMU_o is said to be BCC efficient *if and only if* (a) $b^* = 1$ and (b) all optimum slack values in Eq. (3) are zero. In the case of inefficiency, the following BCC projection of $\Sigma_{j=1}^{n} \lambda_j \leqslant 1$ is BCC efficient:

$$\hat{x}_{io} = b^* x_{io} - s_i^-, \qquad \hat{y}_{ro} = y_{ro} + s_r^+ \tag{5}$$

Proposition 2. *If a DMU is CCR efficient then it is BCC efficient.*

It can be seen that the only difference between the CCR and the BCC models is the convex restriction of $\Sigma_{j=1}^{n} \lambda_j = 1$ in the primal model (u_o in the dual model). If we impose $\Sigma_{j=1}^{n} \lambda_j \leqslant 1$ in the CCR model, then we obtain the following DEA model

$$\begin{aligned} & f^* = \min f \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq f x_{io} \quad i = 1, \ldots, m; \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \ldots, s; \\ & \sum_{j=1}^{n} \lambda_j \leq 1 \\ & \lambda_j \geq 0 \quad j = 1, \ldots, n. \end{aligned} \tag{6}$$

Obviously $\theta^* \leq f^* \leq b^*$. The dual linear programming problem is

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r y_{ro} - u_o \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - u_o \leq 0 \quad j = 1, \ldots, n; \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & u_r,\ v_i,\ u_o \geq 0 \end{aligned} \tag{7}$$

3. Returns to scale

Returns-to-scale (RTS) have typically been defined only for single output situations. Therefore, we generalize the notion of RTS to the multiple output case. Let $\mathbf{T}$ be a production possibility set which is constructed by the inputs and outputs of the n DMUs. Färe *et al.* [18] provide the following definition related to the RTS (see Ref. [13] for an equivalent definition)

> The technology exhibits constant returns to scale (CRS) if $\mu\mathbf{T} = \mathbf{T}$, $\mu > 0$; it exhibits non-increasing returns to scale (NIRS) if $\lambda\mathbf{T} \subseteq \mathbf{T}$, $0 < \lambda \leq 1$; it exhibits non-decreasing returns to scale (NDRS) if $\alpha\mathbf{T} \subseteq \mathbf{T}$, $\alpha \geq 1$ (or equivalently if $\mathbf{T} \subseteq \beta\mathbf{T}$, $0 < \beta \leq 1$).

On the basis of the above RTS definition, we say that a DMU exhibits decreasing returns to scale (DRS) if it exhibits NIRS but not CRS and increasing returns to scale (IRS) if it exhibits neither CRS nor DRS. Furthermore we have the following propositions:

Proposition 3. *The efficient frontier obtained from the CCR model (Eq. (1)) exhibits CRS.*

Proposition 4. *The efficient frontier obtained from the BCC model (Eq. (3)) exhibits variable returns to scale (VRS), i.e. IRS, CRS and DRS are all allowed in the BCC model.*

Proposition 5. *The efficient frontier obtained from the model in Eq. (6) exhibits NIRS.*

On the basis of propositions 1 and 3, we have

Proposition 6. *At least one DMU exhibits CRS.*

We now illustrate these DEA frontiers by a two-dimensional figure with one-input and one-output. Suppose we have six DMUs, A, B, C, D, H and M as shown in Fig. 1. Ray OBC is the CCR efficient frontier exhibiting CRS. AB, BC and CD constitute the BCC efficient frontier and exhibit IRS, CRS and DRS respectively. The efficient frontier obtained from the model in Eq. (6) is constructed by OBC and CD. Obviously IRS are not allowed in the frontier obtained from the model in Eq. (6).

If a DMU is on or projected onto a CCR efficient frontier by the BCC projection (Eq. (5)), then by proposition 3, this DMU exhibits CRS. For instance, the point M in Fig. 1.

Note that some DMUs, say B and C, are located at the intersections of different RTS frontiers. In this situation, CRS have the first priority. That is, B and C should exhibit CRS rather than IRS and DRS, respectively. Thus, on the line segment AB, IRS prevail to the left of B, and on the line segment CD, DRS prevail to the right of C.

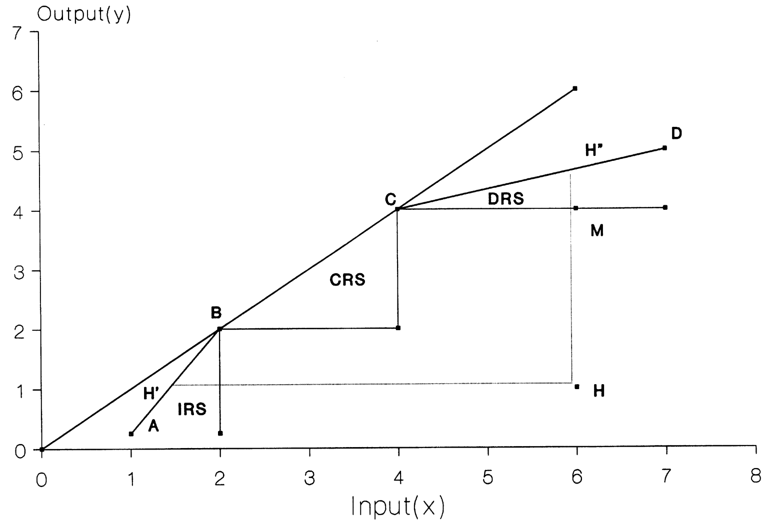

Fig. 1. Frontiers and RTS.

Note also that the concept of RTS may be ambiguous unless a DMU is on the BCC efficient frontier. As mentioned earlier, we classify the RTS for inefficient DMUs by their BCC projections. For instance, by applying the BCC projection (Eq. (5)) to point H, we have a frontier point H′ on the line segment AB and thus H exhibits IRS. However a different RTS classification may be obtained if a different projection (or BCC model) is utilized. This is due to the fact that the input-based and the output-based BCC models yield different projection points on the VRS frontier. For example, the point H is moved onto the line segment CD by the output-based BCC model and thus DRS prevail on the point H″. However some IRS, CRS and DRS regions are uniquely determined no matter which BCC model is employed (see Fig. 1).

To illustrate the discussion to follow we will employ the following sample data set consisting of six DMUs each consuming two inputs to produce a single output (see Table 1). Table 2 provides the efficiency scores obtained from the models in Eqs. (1), (3) and (6). (No non-zero slack values were present.)

Recall that all DMUs can be identified with points E (extreme efficient), E′ (efficient but not an extreme point), F (frontier but not efficient) and N (nonfrontier) [19]. Recall also that the efficient facets are determined by the DMUs in set E. Thus DMUs 1, 2, 3, 5 and 6 belong to set E in the BCC model. The CCR efficient frontier is $x_1+2x_2=6y$ which is constructed by DMUs 1, 2 and 3. By proposition 3, CRS prevail on DMUs 1, 2 and 3. Since DMU4 is projected onto the CRS efficient frontier (i.e. the CCR efficient frontier) by Eq. (5), therefore DMU4 exhibits CRS. DMU5 and DMU6 are, respectively, located on the IRS and DRS efficient frontiers of the BCC model. The RTS classification of each DMU is shown in the last column of Table 2.

4. Theory and methodology of the estimation of RTS

In this section, we first present the three basic (or original) RTS methods and show how these methods work. Next, we indicate the equivalence of these three

Table 1
Sample DMUs

DMU	Input 1 (x_1)	Input 2 (x_2)	Output (y)
1	2	5	2
2	2	2	1
3	4	1	1
4	3	2	1
5	2	1	1/2
6	6	5	5/2

Table 2
Efficiency scores and RTS classifications

DMU	CCR (θ*)	BCC (b*)	Model in Eq. (6) (f*)	RTS
1	1 (E)	1 (E)	1 (E)	CRS
2	1 (E′)	1 (E′)	1 (E′)	CRS
3	1 (E′)	1 (E)	1 (E)	CRS
4	6/7	6/7	6/7	CRS
5	3/4	1 (E)	3/4	IRS
6	15/16	1 (E)	1 (E)	DRS

RTS methods. Finally, we discuss the effect of multiple optima on each of these RTS methods.

4.1. Basic RTS methods

Here we will review the three basic RTS methods in DEA. First consider the scale efficiency index method in which the ratio θ^*/b^* is called scale efficiency index [20].

Theorem 1. *(i) $\theta^* = b^*$ if and only if DMU_o exhibits CRS; otherwise if $\theta^* < b^*$ or equivalently $\theta^* \neq b^*$*, then *(ii) $b^* > f^*$ if and only if DMU_o exhibits IRS; (iii) $b^* = f^*$ if and only if DMU_o exhibits DRS.*

The core of the scale efficiency index method lies in the comparison of different RTS efficient frontiers, respectively, obtained from the models in Eqs. (1), (3) and (6). $\theta^* = b^*$ implies that DMU_o is on or projected onto the CRS frontier, e.g. BC in Fig. 1, which is the intersection of the CCR and BCC efficient frontiers; $b^* > f^*$ (or alternatively $\theta^* = f^*$) implies that DMU_o is on or projected onto the IRS frontier (e.g. AB in Fig. 1) which does not belong to the intersection of efficient frontiers obtained from the models in Eqs. (1) and (6); $b^* = f^*$ implies that DMU_o is on or projected onto the DRS frontier (e.g. CD in Fig. 1) which is the intersection of the efficient frontiers obtained from the BCC model and the model in Eq. (6) but not the model in Eq. (1).

This RTS method can easily be examined by the efficiency scores shown in Table 2. As stated in Ref. [8], this method exploits the natural nesting of the three RTS frontiers and the corresponding ordering of the associated efficiency measures of Eqs. (1), (3) and (6). Note that this method requires solving three DEA models.

On the basis of optimal solutions λ_j^* to Eq. (1), we have the CCR RTS method [4].

Theorem 2. *(i) If $\Sigma_{j=1}^{n}\lambda_j^* = 1$* in any alternate optima then CRS prevail on DMU_o. *(ii) If $\Sigma_{j=1}^{n}\lambda_j^* < 1$* for all alternate optima then IRS prevail on DMU_o. *(iii) If $\Sigma_{j=1}^{n}\lambda_j^* > 1$* for all alternate optima then DRS prevail on DMU_o.

The above RTS method is relative to the concept of most productive scale size (MPSS) (see Ref. [21] for a detailed discussion). Table 3 provides the CCR results with optimal lambda values.

From Table 3, we see that some DMUs, say DMUs 2 and 4, have alternate optimal lambda values. Nevertheless there exists an optimal solution such that $\Sigma_{j=1}^{n}\lambda_j^* = 1$ indicating CRS. DMU5 exhibits IRS because $\Sigma_{j=1}^{n}\lambda_j^* < 1$ in all optima and DMU6 exhibits DRS because $\Sigma_{j=1}^{n}\lambda_j^* > 1$ in all optima.

Let u_o^* represent the optimal value of u_o in the BCC dual model in Eq. (4), then we have the BCC RTS method [5].

Theorem 3. *(i) If $u_o^* = 0$ in any alternate optima then CRS prevail on DMU_o. (ii) If $u_o^* > 0$ in all alternate optima then IRS prevail on DMU_o. (iii) If $u_o^* < 0$ in all alternate optima then DRS prevail on DMU_o.*

Geometrically, for the case of single output, u_o^* represents the y-intercept on the output axis. One can determine RTS classification within BCC RTS method from the optimal u_o values reported in the last column

Table 3
Optimal λ_j and u_o values for the six sample DMUs

DMU	Optimal lambda values in CCR (Eq. (1))	Optimal in BCC (Eq. (4))
1	$\lambda_1^* = 1$; $\Sigma_{j=1}^{6}\lambda_j^* = 1$	[−7, 1]
2	solution 1: $\lambda_2^* = 1$; $\Sigma_{j=1}^{6}\lambda_j^* = 1$ solution 2: $\lambda_1^* = \frac{1}{3}$, $\lambda_3^* = \frac{1}{3}$; $\Sigma_{j=1}^{6}\lambda_j^* = \frac{2}{3}$	[0, 1]
3	$\lambda_3^* = 1$; $\Sigma_{j=1}^{6}\lambda_j^* = 1$	[−5/3, 1]
4	solution 1: $\lambda_1^* = \frac{5}{21}$, $\lambda_3^* = \frac{11}{21}$; $\Sigma_{j=1}^{6}\lambda_j^* = \frac{16}{21}$ solution 2: $\lambda_2^* = \frac{5}{7}$, $\lambda_3^* = \frac{2}{7}$; $\Sigma_{j=1}^{6}\lambda_j^* = 1$ solution 3: $\lambda_1^* = \frac{2}{133}$, $\lambda_2^* = \frac{89}{133}$, $\lambda_3^* = \frac{40}{133}$; $\Sigma_{j=1}^{6}\lambda_j^* = \frac{131}{133}$	[0, 2/7]
5	$0 \leq \lambda_1^* \leq \frac{1}{12}$, $\lambda_2^* = \frac{1}{4} - 3\lambda_1^*$, $\lambda_3^* = \frac{1}{4} + \lambda_1^*$; $\frac{5}{12} \leq \Sigma_{j=1}^{6}\lambda_j^* \leq \frac{1}{2}$	[1/2, 1]
6	$\lambda_1^* \leq \frac{35}{48} - \lambda_2^*/3$, $0 \leq \lambda_2 \leq \frac{35}{16}$, $\lambda_3^* = \frac{25}{24} - \lambda_2^*/3$; $\frac{85}{48} \leq \Sigma_{j=1}^{6}\lambda_j^* \leq \frac{15}{6}$	$(-\infty, -3/37]$

of Table 3. u_o* can take all the optimal u_o values in the intervals $[u_o^-, u_o^+]$. $u_o^* = 0$ is found in DMUs 1, 2, 3 and 4, therefore the four DMUs exhibit CRS. All u_o* are, respectively, bigger and less than zero in DMU5 and DMU6, therefore IRS and DRS, respectively, prevail on DMU5 and DMU6.

Finally, note that the above three basic RTS methods implicitly identify the RTS classifications for BCC inefficient DMUs by their BCC projection (Eq. (5)) (radial component only), i.e. the methods automatically project the BCC inefficient DMUs. Thus, in applications, we need not apply Eq. (5) separately. Note also that some DMUs do have u_o* with a range of both negative and positive values, but they are identified as having CRS, which is consistent with the discussion in the previous section that those DMUs are located on the intersections of CRS and IRS (or DRS) frontiers.

4.2. The relations of the three basic RTS methods

Zhu and Shen [9] showed that the CCR RTS method and the scale efficiency index method are equivalent (see Ref. [7] also). In fact, all the three basic RTS methods are equivalent.

Theorem 4. *(i) $\theta^* = b^*$ if and only if there exists an optimal solution of the CCR primal model (Eq. (1)) with* $\Sigma_{j=1}^n \lambda_j^* = 1$; *(ii)* $\theta^* \neq b^*$ and $b^* > f^*$ if and only if $\Sigma_{j=1}^n \lambda_j^* < 1$ in all optimal solutions of the CCR primal model (Eq. (1)); *(iii)* $\theta^* \neq b^*$ and $b^* = f^*$ if and only if $\Sigma_{j=1}^n \lambda_j^* > 1$ in all optimal solutions of the CCR primal model (Eq. (1)).

Proof. The proof of (i) is obvious from the convex restriction $\Sigma_{j=1}^n \lambda_j = 1$ in the BCC model and hence is omitted.

If $\Sigma_{j=1}^n \lambda_j^* < 1$ in all optimal solutions of the CCR model, then obviously $\theta^* \neq b^*$ (or $\theta^* < b^*$). Furthermore we have $\theta^* = f^*$, thus $b^* > f^*$. This completes the *if* part of (ii).

Note that $\theta^* \neq b^*$ implies that $\Sigma_{j=1}^n \lambda_j = 1$ cannot hold for all optimal solutions to the CCR model. If $b^* = f^*$ then there does not exist any CCR solution such that $\Sigma_{j=1}^n \lambda_j < 1$. (Otherwise we have $\theta^* = f^*$ which conflicts with $\theta^* \neq b^*$.) Therefore $\Sigma_{j=1}^n \lambda_j$ must be greater than one in all CCR optimal solutions. This completes the *only if* part of (iii).

Note that if $\Sigma_{j=1}^n \lambda_j^* > 1$ in all optimal solutions of the CCR model, then θ^* must not be equal to b^*. Thus the *if* part of (iii) follows immediately from the theorems 1 and 2. The *only if* part of (ii) now is obvious. □

Theorem 5. *(i) $\theta^* = b^*$ if and only if there exists an optimal solution of the BCC dual model (Eq. (4)) with $u_o^* = 0$; (ii)* $\theta^* \neq b^*$ and $b^* > f^*$ if and only if $u_o^* > 0$ in all optimal solutions of the BCC dual model (Eq. (4)); *(iii)* $\theta^* \neq b^*$ and $b^* = f^*$ if and only if $u_o^* < 0$ in all optimal solutions of the BCC dual model (Eq. (4)).

Proof. The proof of this theorem is similar to the proof of theorem 4, but based on the dual formulations in Eqs. (2), (4) and (7). □

On the basis of theorems 4 and 5, we have

Corollary 1. *(i) There exists an optimal solution to Eq. (1) with* $\Sigma_{j=1}^n \lambda_j^* = 1$ if and only if there exists an optimal solution to Eq. (4) with $u_o^* = 0$; *(ii)* $\Sigma_{j=1}^n \lambda_j^* < 1$ in all optimal solutions of Eq. (1) if and only if $u_o^* > 0$ in all optimal solutions of Eq. (4); *(iii)* $\Sigma_{j=1}^n \lambda_j^* > 1$ in all optimal solutions of Eq. (1) if and only if $u_o^* < 0$ in all optimal solutions of Eq. (4).

From the above discussion, we observe that the three basic RTS methods are equivalent but provide different ways to estimate RTS for each DMU. The conditions in theorems 2 and 3 (i.e. the CCR and BCC RTS methods) are necessary and sufficient.

4.3. Multiple optima in the primal and dual solutions

It is well known that multiple optimal solutions are likely to occur in the dual formulations of DEA models, e.g. Eq. (4). In addition, the DEA primal model, e.g. the model in Eq. (1), may have alternate optima. As a result, it is probable that the presence of multiple optima in DEA models will affect the RTS estimation. Note that the scale efficiency index method employs the optimal values of three different DEA models, therefore the possible alternate optima should have no effect on this method, i.e. this method is robust even when there are multiple optimal solutions. However, we stress the need to check for possible alternate optima when either one of the other two RTS methods is employed. Unfortunately, in real world applications, the examination of alternative optima is a laborious task and one may attempt to use a single set of resulting optimal solutions in the application of the RTS methods. In the case of the CCR or BCC RTS methods, this may yield erroneous results. For instance, if we obtain $\lambda_1^* = \frac{1}{3}$, $\lambda_3^* = \frac{1}{3}$ or $u_o^* = 1$ for DMU2, then DMU2 may erroneously be classified as having IRS because $\Sigma\lambda_j^* < 1$ or $u_o^* > 0$ in one particular alternate solution.

In Ref. [22], the presence of multiple optima is attributed to the (piecewise) linear production function in DEA. However Zhu and Shen [15] showed that the conclusion in Ref. [22] is erroneous and revealed that linear dependency in a set of CCR efficient DMUs is one cause of multiple optimal lambda values in the CCR model (Eq. (1)), i.e. the DMUs in set E′ will lead to alternate lambda values. Also, DMUs in set F may cause multiple optimal lambda solutions [23, 24]. Relative to the six DMUs in Table 1, we have $DMU_2 = \frac{1}{3}DMU_1 + \frac{1}{3}DMU_3$ in which DMU1 and DMU3 belong to set E. Thus DMU2 belongs to set E′. Therefore, multiple optimal lambda values are detected in DMUs 2, 4, 5 and 6. However, in DMU5, $\Sigma\lambda_j^*$ is always less than one and, in DMU6, $\Sigma\lambda_j^*$ is always greater than one, even if lambda variables have alternate solutions in Eq. (1).

In fact, on the basis of Theorems 1 and 4, we have the following important result

Corollary 2. *(i) If DMU_o exhibits IRS then* $\Sigma_{j=1}^{n}\lambda_j^* < 1$ for all alternate optima in Eq. (1); *(ii) if DMU_o exhibits DRS then* $\Sigma_{j=1}^{n}\lambda_j^* > 1$ for all alternate optima in Eq. (1).

The significance of this corollary lies in the fact that the possible alternate optimal lambda values obtained from Eq. (1) only affect the estimation of RTS for those DMUs that truly exhibit CRS, and have nothing to do with the RTS estimation on those DMUs that truly exhibit IRS or DRS. That is, if a DMU exhibits IRS (or DRS), then $\Sigma\lambda_j^*$ must be less (or greater) than one, no matter whether there exist alternate optima of λ_j.

Turning to the BCC RTS method, note that for DMU5, u_o^* is always bigger than zero and for DMU6, u_o^* is always less than zero. From theorems 1 and 5, we have

Corollary 3. *(i) If DMU_o exhibits IRS then $u_o^* > 0$ for all alternate optima in Eq. (4); (ii) if DMU_o exhibits DRS then $u_o^* < 0$ for all alternate optima in Eq. (4).*

This corollary also indicates that the alternate optimal u_o values obtained from Eq. (4) only affect the determination of RTS on those DMUs that truly exhibit CRS. For example, $-\frac{5}{3} \leq u_o^* \leq 1$ for DMU3, i.e. u_o^* can either be greater than or less than or equal to zero. If we obtain $u_o^* = -5/3$ in an optimal solution, then DMU3 may erroneously be termed as having DRS; if we obtain $u_o^* = 1$, then DMU3 may erroneously be termed as having IRS, while in fact CRS prevail for DMU3.

The above two corollaries are very important in empirical applications because one may erroneously declare a CRS DMU as having IRS or DRS if one does not check all possible optima in the models in Eq. (1) or Eq. (4).

5. The application and usage of the RTS methods

As noted, the possible multiple optimal solutions of lambda or u_o may affect the CCR and BCC RTS methods. However, one is unable to know in advance whether there exist alternate optima in real world applications. Variations or extensions of the two basic RTS methods have been proposed for dealing with multiple optima. Some treat multiple optima directly and some indirectly.

5.1. Treatment of u_o

Banker and Thrall [13] provide two auxiliary linear programming models to deal with the multiple optimal solutions of the free variable, u_o, in the BCC dual model (Eq. (4)), i.e.

$$\begin{aligned} &\max u_o \\ \text{s.t.} \quad &\sum_{r=1}^{s} u_r y_{ro} + u_o = 1 \\ &\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{s} v_i x_{ij} + u_o \leq 0 \quad j = 1, \ldots, n; \qquad (8) \\ &\sum_{i=1}^{m} v_i x_{io} = 1 \\ &u_r,\ v_i \geq 0 \text{ and } u_o \text{ is free} \end{aligned}$$

The optimal value of u_o is u_o^+. Now the objective of Eq. (8) is changed from maximization to minimization. The minimized score of u_o is u_o^-. Obviously we have $u_o^- \leq u_o^* \leq u_o^+$ in which u_o is an optimal solution to Eq. (4). Next u_o^* is transformed into another score ρ_o^* by $\rho_o^* = 1/(1 - u_o^*)$, i.e.

$$\rho_o^- = 1(1 - u_o^-) \leq \rho_o^* = 1(1 - u_o^*) \leq \rho_o^+ = 1(1 - u_o^+)$$

Theorem 6. *On the basis of ρ_o^*, RTS can be identified as CRS if and only if $\rho_o^* = 1$, IRS if and only if $\rho_o^* > 1$ and DRS if and only if $\rho_o^* < 1$.*

Proof. See Ref. [13]. □

The Eq. (8)-like formulations are provided for the calculation of upper and lower boundaries of optimal u_o. It can be seen that the RTS now is determined by intervals. The above method is developed to find out all possible multiple optimal solutions of u_o (see the intervals of u_o^* in the last column of Table 3). $\rho_o^* = 1$, $\rho_o^* > 1$ and $\rho_o^* < 1$, respectively, correspond to $u_o^* = 0$ (in any optima), $u_o^* > 0$ and $u_o^* < 0$ (in all optima). Therefore this method, in fact, is an extension of the BCC RTS method.

Note that Eq. (8) is only valid for the BCC frontier DMUs (i.e. $b^* = 1$). Banker *et al.* [14] show how to identify RTS on each DMU by the BCC projection (Eq. (5)) and how to estimate RTS possibilities from the optimal values for in Eq. (4) without having to examine *all* alternate optima. Nevertheless the following modification makes it possible to characterize the RTS on each DMU without having to use BCC projection given the existence of an optimal solution with $u_o^* > 0$ in Eq. (4), i.e.

$$\begin{aligned} &\min \hat{u}_o \\ \text{s.t.} \quad &\sum_{r=1}^{s} u_r y_{ro} + \hat{u}_o = b^* \\ &\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \hat{u}_o \leq 0 \quad j = 1, \ldots, n; \\ &\sum_{i=1}^{m} v_i x_{io} = 1 \\ &u_r,\ v_i,\ \hat{u}_o \geq 0 \end{aligned} \tag{9}$$

In the case of the solution of the unrestricted variable in Eq. (4) yielding $u_o^* < 0$, we replace $\hat{u}_o \geq 0$ in Eq. (9) by $\hat{u}_o \leq 0$ and change the objective of Eq. (9) from minimization to maximization. Note that if $b^* = 1$, i.e. DMU_o is on the BCC frontier, then Eq. (9) is similar to Eq. (8).

Theorem 7. *(i) Given the existence of an optimal solution with $u_o^* > 0$ in Eq. (4), the RTS at DMU_o are CRS if and only if the optimal value which Eq. (9) achieves is zero, i.e. $\hat{u}_o^* = 0$, and IRS if and only if $\hat{u}_o^* > 0$; (ii) given the existence of an optimal solution with $u_o^* < 0$ in Eq. (4), the RTS at DMU_o are CRS if and only if the optimal value which Eq. (9) achieves is zero, i.e. $\hat{u}_o^* = 0$, and DRS if and only if $\hat{u}_o^* < 0$.*

Proof. Suppose that u_r^*, v_i^* and u_o^* is an optimal solution to Eq. (9) for DMU_o, then $\Sigma_{r=1}^{s} u_r^* y_{ro} + \hat{u}_o^* = b^*$ and $\Sigma_{i=1}^{m} v_i^* x_{io} = 1$. Furthermore $\Sigma_{r=1}^{s} u_r^* y_{ro} + \hat{u}_o^* = b^* \Sigma_{i=1}^{m} v_i^* x_{io}$. Obviously $\Sigma_{r=1}^{s} u_r^* y_r = \Sigma_{i=1}^{m} v_i^* (b^* x_i) - \hat{u}_o^*$ is a supporting hyperplane at $(b^* x_{io},\ i = 1, \ldots, m;\ y_{ro},\ r = 1, \ldots, s)$. From Ref. [13], we can easily obtain that the RTS at DMU_o are CRS *if and only if* $\hat{u}_o^* = 0$, IRS *if and only if* $\hat{u}_o^* > 0$ and DRS *if and only if* $\hat{u}_o^* < 0$. □

5.2. Treatment of λ_j

In order to check alternate optima possibilities in the CCR model, Banker *et al.* [7] provide the following linear programming model (given $\Sigma_{j=1}^{n} \lambda_j^* < 1$ obtained from Eq. (1))

$$\begin{aligned} \max \quad &\sum_{j=1}^{n} \hat{\lambda}_j - \varepsilon \left(\sum_{i=1}^{m} \hat{s}_i^- + \sum_{r=1}^{s} \hat{s}_r^+ \right) \\ \text{s.t.} \quad &\sum_{j=1}^{n} \hat{\lambda}_j x_{ij} + \hat{s}_i^- = \theta_o^* x_{io} \qquad i = 1, 2, \ldots, m; \\ &\sum_{j=1}^{n} \hat{\lambda}_j y_{rj} - \hat{s}_r^+ = y_{ro} \qquad r = 1, 2, \ldots, s; \\ &\sum_{j=1}^{n} \hat{\lambda}_j \leq 1 \\ &\hat{\lambda}_j, \hat{s}_i^-, \hat{s}_r^+ \geq 0 \end{aligned} \tag{10}$$

in which θ_o^* is the optimal θ obtained from Eq. (1) ($\theta_o^* x_{io}$ now are fixed numbers not variables). The optimal solution to Eq. (10) yields the maximum value of $\Sigma_{j=1}^{n} \hat{\lambda}_j^*$. If we obtain an optimal solution with $\Sigma_{j=1}^{n} \lambda_j^* > 1$, then we change the objective of Eq. (10) to minimization and replace $\Sigma_{j=1}^{n} \hat{\lambda}_j \leq 1$ with $\Sigma_{j=1}^{n} \hat{\lambda}_j \geq 1$ and denote the corresponding optimal value of $\Sigma_{j=1}^{n} \hat{\lambda}_j$ by $\Sigma_{j=1}^{n} \hat{\lambda}_j^{**}$. We then have:

Theorem 8. *(i) Given the existence of an optimal solution with* $\Sigma_{j=1}^{n} \lambda_j^* < 1$ in Eq. (1), the RTS at DMU_o are CRS if and only if $\Sigma_{j=1}^{n} \hat{\lambda}_j^* = 1$, and IRS if and only if $\Sigma_{j=1}^{n} \hat{\lambda}_j^* < 1$; *(ii) given the existence of an optimal solution with* $\Sigma_{j=1}^{n} \lambda_j^* > 1$ in Eq. (1), the RTS at DMU_o are CRS if and only if $\Sigma_{j=1}^{n} \hat{\lambda}_j^{**} = 1$, and DRS if and only if $\Sigma_{j=1}^{n} \hat{\lambda}_j^{**} > 1$.

Proof. See Ref. [7]. □

All possible optimal values or intervals of $\Sigma_{j=1}^{n} \lambda_j^*$ can be determined by a Eq. (10)-like formulation, if the value of $\Sigma_{j=1}^{n} \lambda_j^*$ is continuous. For instance, the intervals for $\Sigma_{j=1}^{n} \lambda_j^*$ for DMU5 and DMU6 are, respectively, [5/12, 1/2] and [85/48, 15/6] (see Table 3). It is obvious that theorems 6, 7 and 8 work on the same principle.

5.3. Direct treatment

The above three modified RTS methods (theorems 6, 7 and 8) deal with the multiple optimal solutions in DEA formulations directly in the sense that they try to determine all optimal solutions by intervals and use the upper and lower bounds to estimate the RTS.

Next, we will provide some indirect modifications which try to avoid the effect of multiple optimal DEA solution on RTS estimation. On the basis of corollaries 2 and 3, we know that multiple optimal solutions have nothing to do with the RTS estimation for IRS and DRS DMUs. Therefore we focus on characterizing RTS on the CRS DMUs when we make modifications to the CCR and BCC RTS methods.

By the help of the scale efficiency index method, Zhu and Shen [15] introduce a very simple approach to eliminate the need for examining all alternate optima when employing the CCR RTS method. That is,

Theorem 9. *(i) $\theta^* = b^*$ if and only if CRS prevail on DMU_o; otherwise, if $\theta^* \neq b^*$ then (ii) $\Sigma_{j=1}^{n}\lambda_j^* < 1$* if and only if IRS prevail on DMU_o; *(iii)* $\Sigma_{j=1}^{n}\lambda_j^* > 1$ if and only if DRS prevail on DMU_o.

Thus in empirical applications, we can explore RTS in two steps. First, select all the DMUs that have the same optimal values to Eqs. (1) and (3) regardless of the value of $\Sigma\lambda_j^*$. These DMUs are in the CRS region. Next, use the value of $\Sigma\lambda_j^*$ (in any CCR outcome) to determine the RTS for the remaining DMUs. We observe that in this process we can safely ignore possible multiple optimal solutions of λ_j. For example, although some $\Sigma\lambda_j^*$ are less than one in DMU2 and DMU4, the two DMUs have the same CCR and BCC efficiency scores. Therefore by theorem 9(i), CRS prevail.

Note that this new CCR RTS method requires the solution of the BCC model. However, if $\theta^* = 1$ in Eq. (1), then by proposition 3, we immediately know that CRS prevail. Thus, we only need to solve the BCC model for those DMUs with $\theta^* < 1$ and $\Sigma_{j=1}^{n}\lambda_j^* \neq 1$ in Eq. (1) and then use theorem 9(i). Note also that it is not necessary to compute the optimal values of $\Sigma_{j=1}^{n}\hat{\lambda}_j^*$ and $\Sigma_{j=1}^{n}\hat{\lambda}_j^{**}$ for Eq. (10).

Note that the multiple optimal lambda values are caused by the presence of DMU2 which belongs to set E′. Nevertheless if we exclude DMU2 from the reference set, then DMU4 will be declared as having IRS because the unique solution of $\lambda_1^* = \frac{5}{21}$, $\lambda_3^* = \frac{11}{21}$ and $\Sigma_{j=1}^{6}\lambda_j^* = \frac{16}{21}$. Therefore the CRS of DMU4 is related to the existence of DMU2; removal of DMUs in set E′ (e.g. DMU2) can alter the CRS region.

Similar to theorem 9, we can readily obtain the following results:

Theorem 10. *(i) $\theta^* = b^*$ if and only if CRS prevail on DMU_o; otherwise, if $\theta^* \neq b^*$ then (ii) $u_o^* > 0$ if and only if IRS prevail on DMU_o; (iii) $u_o^* < 0$ if and only if DRS prevail on DMU_o.*

Thus, with the assistance of the CCR efficiency score, possible alternate optimal solutions in Eq. (4) no longer affect the determination of RTS for each DMU. The entire RTS intervals for u_o^* need not be computed, i.e. that $\theta^* = b^*$ includes all instance (if any) where u_o^* has alternate signs and the remaining case ($\theta^* \neq b^*$) must have u_o^*'s of one sign.

We note that among the three basic RTS methods, the only drawback to the scale efficiency index method seems to be the requirement of three computational runs, i.e. solving three DEA models. However this method does have the advantage of being unaffected by alternate optima. Table 4 summarizes the additional requirements of the modifications to the original CCR and BCC RTS methods. This enables us to choose a proper RTS method according to different application situations. For example, if both CCR and BCC models are employed in an application, then we may choose "CCR-2" or "BCC-3". If only the CCR model is employed, then "CCR-1" may be the better choice.

Finally, if we use BCC-like RTS methods, we should note the fact that in real world situations, n (the number of DMUs) is much larger than $m + s$. Consequently, the number of constraints in the DEA dual models, in say, Eqs. (4), (8) and (9), becomes much larger than that in the DEA primal models, in say, Eqs. (1), (3) and (10). For an ordinary linear programming algorithm, there can be a considerable difference between DEA primal and dual models in terms of algorithmic effort and computation time when n becomes very large [24]. For an interesting counterargument the reader is referred to Ref. [25].

6. Concluding remarks

The current paper reviews the three basic RTS method and their modifications. The purpose of this development is to clarify the relations and nature of

Table 4
The comparison of modifications to the CCR and BCC RTS methods

Modification	Source	Additional requirements
CCR-1	Theorem 8 (see also Ref. [7])	calculate two Eq. (10)-like linear programming problems
CCR-2	Theorem 9 (see also Ref. [15])	calculate BCC model (Eq. (3) or Eq. (4))
BCC-1	Theorem 6 (see also Ref. [13])	calculate two Eq. (8)-like linear programming problems
BCC-2	Theorem 7	calculate two Eq. (9)-like linear programming problems
BCC-3	Theorem 10	calculate CCR model (Eq. (1) or Eq. (2))

different existing RTS methods and further to enable DEA users to select a proper RTS method.

It can be seen that, in fact, there exist three basic RTS methods and all the other approaches are modifications developed for the purpose of dealing with multiple optimal solutions in DEA models. It can also be seen that possible alternate optimal solutions only affect the estimation of RTS on DMUs which should be classified as CRS. Also, in terms of computational efforts, the RTS method proposed by Zhu and Shen [15] which is the combination of the CCR RTS method and the scale efficiency method is relatively easy to apply.

On the other hand, the optimal tableau of the BCC model (Eq. (3) or Eq. (4)) provides some information on RTS. In fact, when solving the BCC model for a DMU_o, the reference set (i.e. basis), does not include both IRS and DRS BCC efficient DMUs. As shown [26], there are five categories for the reference set (a facet of the BCC efficient frontier): (a) IRS DMUs; (b) a mixture of IRS and CRS DMUs; (c) CRS DMUs; (d) a mixture of CRS and DRS DMUs and (e) DRS DMUs. Thus, DMU_o should exhibit IRS if reference set belongs to category (a) or (b); CRS if reference set belongs to category (c) and DRS if reference set belongs to category (d) or (e). The above results are obvious in Fig. 1. Therefore it would appear that one only needs to estimate the RTS for the DMUs in set E and all the other DMUs' RTS can be obtained from their reference sets.

Note that the core of the scale efficiency index method is the comparison of CRS, VRS and NIRS frontiers. Thus we may impose $\Sigma_{j=1}^{n}\lambda_j \geq 1$ rather than $\Sigma_{j=1}^{n}\lambda_j \leq 1$ in Eq. (1) and obtain an DEA model satisfying NDRS. As a result, we obtain another scale efficiency index method by using Eqs. (1) and (3) and this new NDRS DEA model [18].

In fact, the RTS of a DMU are strongly related to the positions of the efficient DMUs. For example, in the sample data set, if we adjust DMU3 to (8, 2, 2) while the CCR efficient frontier does not change, IRS will prevail at DMU4. Therefore the sensitivity of RTS would be an interesting topic for further research. The sensitivity issue seems to be related to changes in the efficient frontier (some efficient DMUs become inefficient) and changes of position of the efficient DMUs along the frontiers.

Finally we have discussed RTS estimation in terms of input-based DEA models. Similar developments hold for the output-based DEA models. Furthermore, the RTS results for cone ratio DEA models can be addressed in a similar way without much additional effort (see Ref. [27] for the cone ratio DEA model).

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