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A slack-based measure of efficiency in context-dependent data envelopment analysis

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Abstract

Data envelopment analysis (DEA) has been proven as an excellent data-oriented performance evaluation method when multiple inputs and outputs are present in a set of peer decision-making units (DMUs). In the DEA literature, a context-dependent DEA is developed to provide finer evaluation results by examining the efficiency of DMUs in specific performance levels based upon radial DEA efficiency scores. In DEA, non-zero input and output slacks are very likely to present after the radial efficiency score improvement. Often, these non-zero slack values represent a substantial amount of inefficiency. Therefore, in order to fully measure the inefficiency in DMU's performance, it is very important to also consider the inefficiency represented by the non-zero slacks in the context-dependent DEA. This study proposes a slack-based context-dependent DEA which allows a full evaluation of inefficiency in a DMUs performance. By using slack-based efficiency measure, we obtain different frontier levels and more appropriate performance benchmarks for inefficient DMUs.

Keywords: Slack-based efficiency measure; Context-dependent; DEA; Benchmarking

1. Introduction

Data envelopment analysis (DEA) [1] is a mathematical programming method for evaluating the relative efficiency of decision-making units (DMUs) with multiple outputs and multiple inputs. It is well known that adding or deleting an inefficient DMU does not alter the efficiencies of the existing DMUs and the efficient frontier. The inefficiency scores change only if the efficient frontier is altered. The performance of DMUs depends only on the identified efficient frontier characterized by the DMUs with an unity efficiency score.

If the performance of inefficient DMUs deteriorates or improves, the efficient DMUs still may have a unity efficiency score. Although the performance of inefficient DMUs depends on the efficient DMUs, efficient DMUs are only characterized by an efficiency score of one. The performance of efficient DMUs is not influenced by the presence of inefficient DMUs. However, the evaluation is often influenced by the context. A DMUs performance will appear more attractive against a background of less attractive alternatives and less attractive when compared to more attractive alternatives.

The context-dependent DEA [2,3] is introduced to measure the relative attractiveness of a particular DMU when compared to others. Relative attractiveness depends on the evaluation context constructed from alternative DMUs. The original DEA method evaluates each DMU against a set of

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efficient DMUs and cannot identify which efficient DMU is a better option with respect to the inefficient DMU. This is because all efficient DMUs have an efficiency score of one. Although one can use the super-efficiency DEA model [4–6] to rank the performance of efficient DMUs, the evaluation context changes in each evaluation and the efficient DMUs are not evaluated against the same reference set.

In the context-dependent DEA, the evaluation contexts are obtained by partitioning a set of DMUs into several levels of efficient frontiers. Each efficient frontier provides an evaluation context for measuring the relative attractiveness. When DMUs in a specific level are viewed as having equal performance, the attractiveness measure allows us to differentiate the "equal performance" based upon the same specific evaluation context. A combined use of attractiveness and progress measures can further characterize the performance of DMUs.

The original context-dependent DEA model is developed by using radial efficiency measure, where slack values are not taken into account. If a DMU with an efficiency score of one has non-zero slack value, it is categorized into the same efficiency level together with efficient DMUs in spite that it is inefficient. The slack-based measure (SBM) [7,8] is introduced to evaluate the efficiency based on the slack values. When we use the SBM to evaluate the context, we can have an appropriate stratification of the DMU performance levels. In the current study, we develop a slack-based context-dependent DEA.

The reminder of the paper is organized as follows. The next section briefly introduces the original context-dependent DEA. We then develop the slack-based context-dependent DEA. We then illustrate our proposed DEA method with an example. The final section concludes.

2. Context-dependent data envelopment analysis

Assume that there are n DMUs which produce s outputs by using m inputs. We define the set of all DMUs as J^1 and the set of efficient DMUs in J^1 as E^1 . Then the sequences of J^l and E^l are defined interactively as $J^{l+1} = J^l - E^l$. The set of E^l can be found as the DMUs with optimal value ϕ_o^l of 1 to the following linear programming problem:

$$\begin{aligned}
\underset{\lambda,\phi}{\text{maximize}} \quad & \phi_o^l = \phi \\
\text{subject to} \quad & \sum_{j \in J^l} \lambda_j x_{ij} \leqslant x_{i_o}, \quad i = 1, \ldots, m, \\
& \sum_{j \in J^l} \lambda_j y_{rj} \geqslant \phi y_{r_o}, \quad r = 1, \ldots, s, \\
& \lambda_j \geqslant 0, \;\; j \in J^l,
\end{aligned} \tag{1}$$

where x_{ij} and y_{rj} are ith input and rth output of DMU j. When $l = 1$, model (1) becomes the original output-oriented CCR model (Charnes, Cooper and Rhodes, 1978) and E^1 consists of all the radially efficient DMUs. A radially efficient DMU may have non-zero input/output slack values. The DMUs in set E^1 define the first-level efficient frontier. When $l = 2$, model (1) gives the second-level efficient frontier after the exclusion of the first-level efficient DMUs. In this manner, we identify several levels of efficient frontiers. Then E^l consists the lth level efficient frontier.

Model (1) yields a stratification of the whole set of DMUs, which partitions into different subgroups of efficiency levels characterized by E^l . It is easy to show that these sets of DMUs have the following properties:

(i) $J^1 = \bigcup E^l$ and $E^l \cap E^{l¢} = \emptyset$ for $l \neq l'$;
(ii) The DMUs in $E^{l'}$ are dominated by the DMUs in E^l if $l' > l$;
(iii) Each DMU in set E^l is efficient with respect to the DMUs in set $J^{l¢}$ for all $l' > l$.

Based upon the evaluation context E^l, the context-dependent DEA measures the relative attractiveness of DMUs. Consider a specific DMU$_o$ from a specific level E^l. The following model is used to characterize the attractiveness with respect to levels exhibiting poorer performance in $E^{l¢}$ for $l' > l$.

$$\begin{aligned}
\underset{\lambda,\phi}{\text{maximize}} \quad & \phi_o^{l¢} = \phi \\
\text{subject to} \quad & \sum_{j \in J^{l¢}} \lambda_j x_{ij} \leqslant x_{i_o}, \quad i = 1, \ldots, m, \\
& \sum_{j \in J^{l¢}} \lambda_j y_{rj} \geqslant \phi y_{r_o}, \quad r = 1, \ldots, s, \\
& \lambda_j \geqslant 0, \;\; j \in J^{l¢}.
\end{aligned} \tag{2}$$

It is easy to show that $\phi_o^{l¢} < 1$ for $l' > l$, and $\phi_o^{l_1} < \phi_o^{l_2}$ for $l_1 > l_2$. Then $A_o^{l¢} \equiv 1/\phi_o^{l¢}$ is called the (output oriented) attractiveness of DMU$_o$ from a specific level $E^{l¢}$. In model (2), each efficient frontier represents an evaluation context for evaluating the relative attractiveness of DMUs in $E^{l¢}$. Note that the larger the value of $A_o^{l¢} > 1$, the more attractive DMU$_o$ is, because DMU$_o$ makes itself more distinctive from the evaluation context $E^{l¢}$. We are able to rank the DMUs in $E^{l¢}$ based upon their attractiveness scores and identify the best one.

To improve the performance of an inefficient DMU, the target of improvement should be given among the efficient DMUs. The reference set suggests the target of improvement for the inefficient DMUs. Actually, when $l = 1$, model (1) gives the reference set of DMUs from the efficient DMUs for inefficient DMUs. It may be a final goal of improvement, however, for some inefficient DMUs, this goal may be quite different from the current performance and difficult to achieve. Therefore, it is not appropriate to set an improvement benchmark target based upon the efficient DMUs directly. Step-by-step improvement is a useful alternative, and the benchmark target at each step is provided based on the evaluation context at each efficient frontier level.

3. Slack-based context-dependent DEA

The above context-dependent DEA is developed by using a radial efficiency measure, which ignores possible non-zero slack values. A SBM of efficiency [8] is introduced to evaluate the efficiency together with the slack value. The following index ρ

$$\rho = \frac{1 - 1/m\sum_{i=1}^{m} s_i^-/x_{i_o}}{1 + 1/s\sum_{r=1}^{s} s_r^+/y_{r_o}} \tag{3}$$

is defined in terms of the amount of slack, and has the value between 0 and 1. The SBM efficiency score is obtained from the following linear program,

$$\begin{aligned}
\text{Minimize} \quad & \rho = \frac{1 - 1/m\sum_{i=1}^{m} s_i^-/x_{i_o}}{1 + 1/s\sum_{r=1}^{s} s_r^+/y_{r_o}} \\
\text{subject to} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{i_o},\ i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{r_o},\ r = 1, \ldots, s, \\
& \lambda_j \geqslant 0,\ j = 1, \ldots, n, s_i^- \geqslant 0,\ i = 1, \ldots, m, \\
& s_r^+ \geqslant 0, r = 1, \ldots, s.
\end{aligned} \tag{4}$$

The SBM efficiency score is less than CCR efficiency score, and CCR inefficient DMU never become SBM efficient. The SBM efficiency score is normalized between 0 and 1, and we have that if, and only if, $\rho^* = 1$, then it is efficient, because $\rho^* = 1$ implies that all slacks are zero and the DMU locates on the efficient frontier. The slack-based score is units invariant.

Based upon (4), we propose a stratification procedure in the same manner to the original context-dependent DEA described in Section 2 as $J^{l+1} = J^l - E^l$, where the set of efficient DMUs E^l is determined from the slack-based efficiency score. That is, the set of E^l can be found as the DMUs with optimal value ρ_o^l of 1 to the following programming problem:

$$\begin{aligned}
\text{Minimize} \quad & \rho_o^l = \frac{1 - 1/m\sum_{i=1}^{m} s_i^-/x_{i_o}}{1 + 1/s\sum_{r=1}^{s} s_r^+/y_{r_o}} \\
\text{subject to} \quad & \sum_{j \in J^l} \lambda_j x_{ij} + s_i^- = x_{i_o}, \quad i = 1, \ldots, m, \\
& \sum_{j \in J^l} \lambda_j y_{rj} - s_r^+ = y_{r_o}, \quad r = 1, \ldots, s, \\
& \lambda_j \geqslant 0,\ j \in J^l, s_i^- \geqslant 0,\ i = 1, \ldots, m, \\
& s_r^+ \geqslant 0, r = 1, \ldots, s.
\end{aligned} \tag{5}$$

The intensity vector λ in (5) shows the reference set $R_o^{\text{SBM}}(l)$ of DMU$_o$ in the efficiency level k based on the context E^l for $l < k$.

$$R_o^{\text{SBM}}(l) = \{j \in J^l | \lambda_j > 0 \text{ in } (5)\}. \tag{6}$$

For context-dependent DEA by radial efficiency measure, the reference set $R_o^{\text{CCR}}(l)$ of DMU$_o$ in the efficiency level k based on the context E^l for $l < k$ is similarly given by

$$R_o^{\text{CCR}}(l) = \{j \in J^l | \lambda_j > 0 \text{ in } (2)\}. \tag{7}$$

Note that the reference set based on the context E^1 is the same as the reference set of slack-based DEA model. The DMUs in the reference set can be used as benchmark targets for inefficient DMU. The context-dependent DEA provides several benchmark targets by setting evaluation context. To improve efficiency, we use step by step benchmark targets which are provided according to the efficiency level that the DMU exists. The sequence of reference sets $R_o(l),\ R_o(l-1), \ldots, R_o(1)$ is used as the step by step benchmark targets for DMU$_o$.

Now, the attractiveness based on the evaluation context E^l is measured with respect to the DMUs in the subset J^l. For example, the attractiveness for DMU$_o$ based on the evaluation context E^l is obtained from the following programming problem.

$$\begin{aligned}
\text{minimize} \quad & \delta = \frac{1/m\sum_{i=1}^{m} \bar{x}_i/x_{i_o}}{1/s\sum_{r=1}^{s} \bar{y}_r/y_{r_o}} \\
\text{subject to} \quad & \bar{x}_i \geqslant \sum_{j \in J^l} \lambda_j x_{ij}, \quad i = 1, \ldots, m, \\
& \bar{y}_r \leqslant \sum_{j \in J^l} \lambda_j y_{rj}, \quad r = 1, \ldots, s, \\
& \lambda_j \geqslant 0,\ j \in J^l, \bar{x}_i \geqslant x_{io},\ i = 1, \ldots, m, \\
& 0 \leqslant \bar{y}_r \leqslant y_{r_o},\ r = 1, \ldots, s.
\end{aligned} \tag{8}$$

Note that this fractional programming problem can be transformed into a linear programming problem using the Charnes–Cooper transformation [7] as

$$\begin{aligned}
\text{minimize} \quad & \tau = \frac{1}{m}\sum_{i=1}^{m} \frac{\tilde{x}_i}{x_{i_o}} \\
\text{subject to} \quad & \frac{1}{s}\sum_{r=1}^{s} \frac{\tilde{y}_r}{y_{r_o}} = 1, \\
& \tilde{x}_i \geqslant \sum_{j \in J^l} \Lambda_j x_{ij}, \quad i = 1, \ldots, m, \\
& \tilde{y}_r \leqslant \sum_{j \in J^l} \Lambda_j y_{rj}, \quad r = 1, \ldots, s, \\
& \tilde{x}_i \geqslant t x_{i_o},\ i = 1, \ldots, m,\ 0 \leqslant \tilde{y}_r \leqslant t y_{r_o}, \\
& r = 1, \ldots, s,\ t > 0, \Lambda_j \geqslant 0,\ j \in J^l.
\end{aligned} \tag{9}$$

Let an optimal solution of (9) be $(\tau^*, \tilde{x}^*, \tilde{y}^*, \Lambda^*, t^*)$. Then we have an optimal solution of (8) as:

$$\delta^* = \tau^*, \lambda^* = \Lambda^*/t^*, \overline{x}^* = \tilde{x}^*/t^*, \overline{y}^* = \tilde{y}^*/t^*. \quad (10)$$

The slack-based attractiveness and slack-based super-efficiency score [7] share some similarity. Consider the subset $\overline{P}_{-o}$ of the production possibility set excluding DMU$_o$.

$$\overline{P}_{-o} = \left\{ (\overline{x}, \overline{y}) \left| \overline{x} \geqslant \sum_{j \neq o} \lambda_j x_j, 0 \leqslant \overline{y} \leqslant \sum_{j \neq o} \lambda_j y_j, \lambda \geqslant 0 \right. \right\} \cap \{(\overline{x}, \overline{y}) | \overline{x} \leqslant x_o, 0 \leqslant \overline{y} \leqslant y_o\}. \quad (11)$$

The L_1 distance between (x_o, y_o) and $\overline{P}_{-o}$ is used as the index of slack-based super-efficiency, which is obtained from the following programming problem.

$$\text{Maximize} \quad \frac{1/m \sum_{i=1}^{m} \overline{x}_i / x_{i_o}}{1/s \sum_{r=1}^{s} \overline{y}_r / y_{r_o}}$$

$$\text{subject to} \quad \overline{x}_i \geqslant \sum_{j \neq o} \lambda_j x_{ij}, \; i = 1, \ldots, m,$$

$$\overline{y}_r \leqslant \sum_{j \neq o} \lambda_j y_{rj}, \; r = 1, \ldots, s,$$

$$\lambda_j \geqslant 0, \; j \neq o, \overline{x}_i \geqslant x_{i_o}, \; i = 1, \ldots, m,$$
$$0 \leqslant \overline{y}_r \leqslant y_{r_o}, \; r = 1, \ldots, s. \quad (12)$$

Note that in slack-based super-efficiency, the evaluation context (reference set) changes for each efficient DMU. Our slack-based context-dependent DEA uses the same evaluation context (reference set).

4. Application

A power company in Osaka, Japan has 14 sales branches, and each sales branch does business independently in its covered district. When a business sales branch having poor performance tried to improve its performance by benchmarking the best business sales branches, it was very difficult to do so, Because there was a huge performance gap between the best practices and the underperforming sales branches. This indicates that it is necessary to provide an attainable benchmark target via a step-wise improvement.

This company measures the performance of sales branches in terms of four indices, namely management, mobility, planning and presentation. Table 1 shows the data of 14 sales branches. The purpose of the analysis is to find the good performer, and to provide an appropriate benchmark target for poor performers.

We apply the slack-based context-dependent DEA, where each sales branch is viewed as a DMU and the four indices are used as four outputs with one input of unity. Table 1 also reports the radial and slack-based efficiency scores for each sales branch. The corresponding efficiency level is calculated by using the context-dependent DEA model. By the stratification of sales branches, we have five efficiency levels for both models. A remarkable difference is found in the sales branches with non-zero slack value. For example, Branch L has a radial efficiency score of 1, however, it is not efficient because of the non-zero slacks in indices of management, mobility and planning. The slack-based efficiency score reflects the amount of non-zero slack, and the efficiency score is less than 1. Branch L is categorized into level 2 by slack-based context-dependent DEA.

The reference sets for benchmark targets $R^{\text{CCR}}(l)$ by radial context-dependent DEA and $R^{\text{SBM}}(l)$ by slack-based context-dependent DEA are shown in Tables 2 and 3,

Table 1
Index value, efficiency scores and efficiency levels for 14 branches

Sales branch	Management	Mobility	Planning	Presentation	Radial measure		Slack-based measure	
					θ	Level	ρ	Level
A	14.1	25.4	4.9	4.4	0.758	2	0.352	3
B	13.2	20.5	6.4	4.7	0.465	4	0.423	4
C	11.5	16.5	4.0	4.4	0.402	5	0.162	5
D	15.6	23.6	6.4	7.4	0.728	3	0.558	3
E	15.2	18.8	6.4	0.8	0.681	4	0.101	4
F	15.2	22.4	9.0	6.0	0.681	2	0.626	2
G	14.7	23.9	9.1	8.3	0.804	2	0.698	2
H	18.0	29.0	11.3	10.2	1.000	1	1.000	1
I	16.4	23.5	6.3	7.9	0.813	2	0.568	2
J	18.1	26.4	13.0	10.0	1.000	1	1.000	1
K	10.4	20.0	4.8	8.6	0.835	3	0.247	4
L	15.1	25.6	8.8	10.2	1.000	1	0.868	2
M	12.9	16.9	7.9	7.4	0.711	3	0.375	4
N	12.8	20.4	8.8	8.7	0.845	2	0.543	3

Table 2
Reference sets as benchmark targets by radial context-dependent DEA

Branch	Level	$R^{CCR}(1)$	$R^{CCR}(2)$	$R^{CCR}(3)$	$R^{CCR}(4)$
A	2	H			
B	4	H (62%), J (38%)	G (46%), I (43%), A (10%)	D (73%), M (27%)	
C	5	J	I (99%), A (1%)	D	B
D	3	J (64%), H (36%)	I (67%), G (19%), A (14%)		
E	4	J	I (81%), F (19%)	D (94%), M (6%)	
F	2	J (90%), H (10%)			
G	2	H			
H	1				
I	2	J			
J	1				
K	3	H	N		
L	1				
M	3	H	G (91%), N (9%)		
N	2	H			

Table 3
Reference sets as benchmark targets by slack-based context-dependent DEA

Branch	Level	$R^{SMB}(1)$	$R^{SMB}(2)$	$R^{SMB}(3)$	$R^{SMB}(4)$
A	3	J	L		
B	4	J	L	D (86%), N (14%)	
C	5	J	L	N	M
D	3	J	L (62%), I(38%)		
E	4	H	L (92%), I(8%)	D (86%), N (14%)	
F	2	J			
G	2	J			
H	1				
I	2	J			
J	1				
K	4	J	L	N	
L	2	H			
M	4	J	L	N	
N	3	J	L		

respectively. If there are plural DMUs in the reference set, the referred percentage is shown in the parentheses. Table 2 tells us that, for example, the benchmark targets of Branch C are found as Branch J on level 1, Branches I and A on level 2, Branch D on level 3 and Branch B on level 4. Since Branch C is on level 5, it is not a good idea to benchmark Branch J on level 1 directly. At the first step, Branch B on level 4 should be benchmarked. Note that although Branch L is in level 1 by the radial context-dependent DEA, it is not benchmarked by other branches because of the non-zero slacks. However, Branch L's performance is excellent and should be used as a benchmark for inefficient branches. Actually, by the slack-based context-dependent DEA, Branch L is benchmarked from branches in level 3 and more based on the second context.

We now turn to the attractiveness scores for the branches in the first and second levels, which are reported in Table 4 based on radial context-dependent DEA and in Table 5 based on slack-based context-dependent DEA. There are two efficient branches (Recall that Branch L has non-zero slack value). On both of the evaluation contexts E^2 and E^3, Branch J is the best branch because it has the largest attractiveness score.

Consider the sales branches in the first and second levels on the evaluation context E^3. We can rank them in the order of J, H, L, G, F and I by slack-based context-dependent DEA. This ranking is based on the context E^3, so the sales branches in the higher level are not always ranked higher position than those in the lower levels. Moreover, Branch L is ranked in the third position by radial as well as slack-based context-dependent DEA, which indicate that the performance of Branch L is good. It is reasonable that Branch L is used as a benchmark by less efficient branches, which can be realized by the slack-based

Table 4
Attractiveness scores for the sales branches in the first and second levels by radial context-dependent DEA

Level	Branch	Evaluation context	
		E^2	E^3
1	H	1.242	1.523
	J	1.429	1.646
	L	1.190	1.319
2	F	1	1.200
	G	1	1.237
	I	1	1.062
	A	1	1.076
	N	1	1.162

Table 5
Attractiveness scores for the sales branches in the first and second levels by slack-based context-dependent DEA

Level	Branch	Evaluation context	
		E^2	E^3
1	H	1.143	1.313
	J	1.149	1.322
2	L	1	1.144
	F	1	1.061
	G	1	1.078
	I	1	1.029

context-dependent DEA, not the radial context-dependent DEA.

5. Concluding remarks

We have demonstrated the benefits when the context-dependent DEA is derived by using slack-based measure, where the evaluation context is constructed taking the non-zero slack value into account. The introduction of slack-based measure is a slight change in the attractiveness score, but the benchmark target is appropriately provided.

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