Cooperative advertising, game theory and manufacturer–retailer supply chains

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Abstract

Cooperative (co-op) advertising plays a significant role in marketing programs in conventional supply chains and makes up the majority of promotional budgets in many product lines for both manufacturers and retailers. We develop three strategic models for determining equilibrium marketing and investment effort levels for a manufacturer and a retailer in a two-member supply chain. Especially, we address the impact of brand name investments, local advertising, and sharing policy on co-op advertising programs in these models. The first model offers a formal normative approach for analyzing the traditional co-op advertising program where the manufacturer is the leader and the retailer is a follower. The second model provides a further analysis on this manufacturer-dominated relationship. The third model incorporates the recent market trend of retailing power shifts from manufacturers to retailers to analyze efficiencies of co-op advertising programs. We examine the effect of supply chain on the differences in profits resulting from following coordinated strategies as opposed to leader–follower strategies. A cooperative bargaining approach is utilized for determine the best co-op advertising scheme for achieving full coordination in the supply chain.

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1. Introduction

Cooperative (co-op) advertising is an interactive relationship between two members in a manufacturer–retailer supply chain. In this relationship, the retailer initiates and implements a local advertisement and the manufacturer pays part of the costs. A manufacturer uses co-op advertising to motivate immediate sales at the retail level and uses national advertising to influence potential consumers to consider its brand and to help develop brand knowledge and preference. Retailer’s local advertising, with the passage of time, brings potential consumers to the stage of desire and action and gives an immediate reason to buy. Co-op advertising offers consumers the information needs when they move through the final stages of purchase and a congruence of information and information needs that would be impossible if the manufacturer uses only national advertising. In addition to the same objective of immediate sales at the retail level as the manufacturer, the retailer utilizes co-op advertising to reduce substantially its total promotional expense by sharing the cost of advertising with the manufacturer.

In the marketing and economics literature, co-op advertising models in manufacturer–retailer supply chains have focused on a relationship where the manufacturer is a leader and the retailer is a follower. The main subject of the research is design and management [?–?]. We are intended to discuss the relationship between classical co-op advertising models and fully coordinated co-op advertising models developed in this paper. The investigation of the
interactive relationship between a manufacturer and a retailer in a supply chain involves three scenarios. The first scenario utilizes a game structure that formulates the interactive relationship between the manufacturer and the retailer as a non-cooperative, and two-stage game, the second scenario deals with a newly developed higher order two-stage game structure, and the third scenario develops a fully coordinated game structure.

The retailer’s sales response volume function of the product, \( S \), is assumed to be affected mainly by the retailer’s local advertising level, \( a \), and the manufacturer’s national brand name investments, \( q \), which include national advertising and control of implementing co-op advertising agreement between the manufacturer and the retailer. As Young and Greyser point out that co-op advertising is used to attract the attention of customers near the time of actual purchase and therefore it is to stimulate short-term sales. The manufacturer’s brand name investments such as the national advertising is intended to take the potential customers from the awareness of the product to the purchase consideration.

The function of the local advertising is to bring potential customers to the stage of desire and action, to give reasons such as low price and high quality to buy, and to state when and where to obtain the product. Therefore, the manufacturer’s brand name investments and the retailer’s local advertising perform different but complementary functions which have positive effects on the ultimate product sales. Saturation may be reached when both or either the local advertising efforts and the brand name investments are increased. Since co-op advertising is intended to generate short-term sales, we may consider one-period sales response volume function as \( S(a, q) = 1 - a^{-\gamma}q^{-\delta} \), where \( \gamma \) and \( \delta \) are positive constants. There is a substantial literature on the estimation of sales response volume functions (see, for example [?]), but all of them consider only the local advertising effect, not others such as national advertising on the volume of sales.

The manufacturer’s dollar marginal profit for each unit to be sold is \( \rho_m \), and the retailer’s dollar marginal profit is \( \rho_r \).

The fraction of total local advertising expenditures which manufacturer agrees to share with retailer is \( t \), which is the manufacturer’s co-op advertising reimbursement policy.

The manufacturer’s, retailer’s and system’s profit functions are as the following:

\[
\pi_m = \rho_m(1 - a^{-\gamma}q^{-\delta}) - ta - q, \quad (1)
\]

\[
\pi_r = \rho_r(1 - a^{-\gamma}q^{-\delta}) - (1 - t)a, \quad (2)
\]

\[
\pi = \pi_m + \pi_r = (\rho_m + \rho_r)(1 - a^{-\gamma}q^{-\delta}) - a - q. \quad (3)
\]

This paper is organized in the following manner. In Section 2, the relationship between the manufacturer and the retailer as an interactive two-stage game is developed and formulated. In this interactive two-stage co-op advertising structure, the manufacturer, as the leader, first specifies the brand name investments and the co-op reimbursement policy. The retailer, as the follower, then decides on the local advertising level. The Stackelberg equilibrium is achieved.

In Section 3, we discuss higher order Stackelberg equilibrium of the two-stage game model. An analysis indicates that if the manufacturer (leader) moves its strategy away from Stackelberg equilibrium, its profit will be higher than before. Since the retailer is rational, the change of the manufacturer’s strategy forces him/her to reconsider its own strategy. Therefore, a new equilibrium, i.e., higher order Stackelberg equilibrium can be achieved. We show that, both members are better off at higher order Stackelberg equilibrium than at Stackelberg equilibrium.

In Section 4, we relax the leader–follower structure by assuming a symmetric relationship between the manufacturer and the retailer. We focus on the discussion of fully coordinated models for co-op advertising. We show that, (i) all Pareto efficient co-op advertising schemes are associated with a single local advertising level and a single brand name investment quantity; (ii) among all possible co-op advertising schemes, the system profit (the sum of the manufacturer’s and the retailer’s profits) is maximized for every Pareto efficient scheme, but not for any other schemes; (iii) the system profit at any Pareto efficient scheme is higher than at both Stackelberg and higher order Stackelberg equilibriums; (iv) the manufacturer’s brand name investments at full coordination is higher than at Stackelberg equilibrium but lower than at higher order Stackelberg equilibrium; (v) the local advertising level at full coordination is higher than at both Stackelberg and higher order Stackelberg equilibriums, and (vi) there is a subset of Pareto efficient co-op advertising schemes on which both the manufacturer and the retailer achieve higher profits than at both Stackelberg and higher order Stackelberg equilibriums and which are determined by the sharing policy of the local advertising expenditures between the manufacturer and the retailer.

Among those feasible Pareto efficient co-op advertising schemes, the question is which one is the best (sharing policy) for both system members. We address this issue and consider one cooperative bargaining model for determining the best sharing policy. Preferences of the manufacturer and the retailer for the amount of shares of the system profit gain are represented by their cardinal (especially additive cardinal) utilities. Managerial implications of bargaining results are discussed. Concluding remarks are in Section 5. All proofs of results are in Appendix A.

2. Interactive two-stage co-op advertising model

We model the relationship between the manufacturer and the retailer as an interactive two-stage game with the manufacturer as the leader and the retailer as the follower. The solution of the game is called Stackelberg equilibrium. The original idea of co-op advertising came from the demands of the retailer’s promotional help from the manufacturer in order to increase the retailer’s advertising budgets without spending more of retailer’s own funds. In the absence of the manufacturer’s co-op advertising funds, the retailer
will usually spend less money on the local advertising than the amount that is optimal from the manufacturer’s point of view. The manufacturer can use co-op advertising subsidization policy to induce the retailer to increase its local advertising expenditure at a level that results in additional sales of the product to the retailer and, thereby, to the manufacturer. The determination of the level of local advertising expenditures depends on how much the manufacturer is willing to subsidize the retailer. The manufacturer may set up some requirements on the co-op advertising such as the size of the advertisement, the display of the manufacturer’s brand name, and certain product features. The manufacturer, as the leader, first declares the level of brand name investments and the co-op advertising policy. The retailer, as the follower, then decides on the quantity of products to be purchased from the manufacturer taking into account the total local advertising spending to be spent. In other words, the retailer takes the (Stackelberg) equilibrium local advertising expenditures into account in deciding the volume of product to be ordered. The manufacturer on the other hand maximizes its profits by specifying the level of brand investments and co-op advertising reimbursement taking the behavior of the retailer into account.

We first solve for the reaction functions in the second stage of the game. Since \( \pi_t \) is a concave function of \( a \), the optimal value of the local advertising expenditures is determined by setting the first derivative of \( \pi_t \) with respect to \( a \) to be zero as follows:

\[
a = \left( \frac{\gamma p_t \beta}{(1 - t)q^0} \right)^{1/(\gamma + 1)}. \tag{4}
\]

Eq. (4) describes positive and negative changes in the corresponding to the changes in the manufacturer’s co-op advertising reimbursement policy and brand name investments. These can be seen by observing that

\[
\frac{\partial a}{\partial t} = \frac{1}{\gamma + 1}(\gamma p_t/q^0)^{1/(\gamma + 1)}(1 - t)^{-(2 + \gamma)/(\gamma + 1)} > 0, \tag{5}
\]

\[
\frac{\partial a}{\partial q^0} = -\frac{\delta}{\gamma + 1}(\gamma p_t/(1 - t))^{1/(\gamma + 1)}q^{-(\gamma + \delta + 1)/(\gamma + 1)} < 0. \tag{6}
\]

Few managerial implications of (4) and (5) can be obtained as follows: First, the more the manufacturer is willing to share the cost of local advertising, the more the retailer will spend on the local advertising. Therefore, the manufacturer’s co-op advertising policy can be used as an indicator of the amount of money that the retailer would spend on local advertising. The manufacturer can use this indicator to induce the retailer to increase local advertising expenditure at a level that the manufacturer expects. Second, greater manufacturer’s share of local advertising spending would lead to more retailer’s spending for local advertising with the ultimate result of increased sales for both the retailer and the manufacturer. Finally, if the retailer is to capitalize effectively on the brand awareness created by the manufacturer’s national advertising, he/she might have to increase or decrease local advertising expenditures in accordance with the effect of national advertising. In other words, the retailer has strong incentive to spend less money on the local advertising if the manufacturer increases the level of brand name investments. The amount of money spent by the manufacturer on national advertising can be used as another indicator of the amount of money that the retailer would spend on local advertising.

Next, the optimal values of \( q \) and \( t \) are determined by maximizing the manufacturer’s profit subject to the constraint imposed by (4). Therefore, the manufacturer’s optimal problem can be formulated as

\[
\text{Max } \pi_m = \rho_m(1 - q^{-\delta}) - ta - q \tag{7}
\]

subject to

\[
0 \leq t \leq 1, \quad q \geq 0.
\]

where

\[
a = \left( \frac{\gamma p_t}{(1 - t)q^0} \right)^{1/(\gamma + 1)}.
\]

Substituting

\[
a = \left( \frac{\gamma p_t}{(1 - t)q^0} \right)^{1/(\gamma + 1)}
\]

into the objective yields the following problem:

\[
\text{Max } \pi_m = \rho_m \left[ 1 - (\gamma p_t)q^{-\delta}(1 - t)^{\gamma/(\gamma + 1)} - a \right] - (\gamma p_t)q^{-\delta}(1 - t)q^{-\delta(\gamma + 1)} - q
\]

subject to

\[
0 \leq t \leq 1, \quad q \geq 0.
\]

\[
\text{Theorem 1. Let}
\]

\[
a^* = \left( \delta^{-\delta,\gamma,\beta} (\rho_m - \gamma p_t) \right)^{1/(\delta + \gamma + 1)}, \tag{9}
\]

\[
n^* = (\rho_m - (\gamma + 1)p_t)/(\rho_m - \gamma p_t), \tag{10}
\]

\[
q^* = \left( \delta^{-\delta,\gamma,\beta} (\rho_m - \gamma p_t) \right)^{1/(\delta + \gamma + 1)}. \tag{11}
\]

Then \((a^*, n^*, q^*)\) is the equilibrium point of the two-stage game.

From the above optimal formulations, the fraction level \( n^* \) is positively and negatively correlated to changes in manufacturer’s marginal profits and retailer’s marginal profits, respectively,

\[
\frac{\partial n^*}{\partial \rho_m} = \frac{\rho_t}{(\rho_m - \gamma p_t)^2} > 0 \quad \text{and} \quad \frac{\partial n^*}{\partial p_t} = -\frac{\rho_m}{(\rho_m - \gamma p_t)^2} < 0.
\]

For manufacturer, if his marginal profit is high (for instance, those manufacturers who produce infrequently purchased good such as appliances and linens), he/she knows that infrequently purchased products are not very standing out most noticeably to most consumers, except at the time of purchase or need. Once consumer decides to purchase...
this kind of product, one always or often makes an overt search among local sources of information, seeking specific product information. In order to give the retailer more incentive to attract consumers, the manufacturer should share more local advertising expenditures with the retailer. On the other hand, if retailer’s marginal profit is high, at this situation retailer has strong incentive to spend money in local advertising to attract consumers to buy these products, even though the manufacturer only shares a small fraction of local advertising expenditures [?].

3. Higher order Stackelberg equilibrium

Now let us consider both the manufacturer’s and the retailer’s profit functions (??) and (??) again. Given a value of \( a \), Eqs. (??) and (??) can be used to describe the manufacturer’s and the retailer’s iso-profit curves \( \pi_m \) and \( \pi_r \), respectively, in the \((q,t)\) space. Let \( a = a^* \), both the manufacturer’s and the retailer’s iso-profit curves \( \pi^*_m \) and \( \pi^*_r \) pass through \((q^*,t^*)\) which corresponds to the manufacturer’s optimal strategy at the two-stage game structure (see Fig. ??).

For given values of \( \pi_m \) and \( \pi_r \), the associated iso-profit curves are shown in Fig. ??, The manufacturer (retailer) is indifferent between any two sets of \((q,t)\) which lie on the iso-profit curve \( \pi^*_m \). The retailer is indifferent between any two iso-profit curves, the one further away from the origin in Fig. ?? has a higher value of \( \pi_r \). Similarly, for any given retailer’s two iso-profit curves, the one further away from the origin in Fig. ?? has a higher value of \( \pi_r \).

There are several implications of Fig. ??, First, the manufacturer is indifferent between setting up \((q,t) = (q^*, t^*)\) and any other points which lie on the iso-profit curve \( \pi^*_m \). The retailer is indifferent between the manufacturer’s strategy \((q,t) = (q^*, t^*)\) and any other combinations of \( q \) and \( t \) which lie on the iso-profit curve \( \pi^*_r \). Second, the manufacturer prefers all combinations of \( q \) and \( t \) which lie in the region below the iso-profit curve \( \pi^*_m \). The retailer prefers all combinations of \( q \) and \( t \) which lie in the region above the iso-profit curve \( \pi^*_r \). Third, the manufacturer prefers not to consider any combination of \( q \) and \( t \) which lie in the region above the iso-profit curve \( \pi^*_m \), and the retailer prefers not to see the manufacturer’s consideration of any combinations of \( q \) and \( t \) which lie in the region below the iso-profit curve \( \pi^*_r \).

From Eqs. (??) and (??), both the manufacturer’s and retailer’s iso-profit curves \( \pi^*_m \) and \( \pi^*_r \) pass through \((q^*, t^*)\) for given \( a = a^* \). These two curves cannot be tangent to each other at \((q^*, t^*)\), since the gradients of both \( \pi_m \) and \( \pi_r \) are not parallel at \((q^*, t^*)\) for given \( a = a^* \). Hence, there always exist some feasible values of \( q \) and \( t \) which are above the retailer’s

\[
\pi(t) = (\pi_r - \pi_m)/a + \pi_r a^{-(\gamma+1)} q^{-\delta} + 1.
\]

Since \( \pi_m \) and \( \pi_r \) are concave functions of \( q \) and \( t \), the two \((q,t)\) sets satisfying \( \pi_m(1 \times q^{-\delta}) - t a \leq q \geq \pi_m(1 - a^{-\delta} q^{-\delta}) - (1 - t) a \geq \pi_r \) are convex sets for given values of \( \pi_m \) and \( \pi_r \). Therefore, for any given manufacturer’s two iso-profit curves, the one closer to the origin in Fig. ?? has a higher value of \( \pi_m \). Similarly, for any given retailer’s two iso-profit curves, the one further away from the origin in Fig. ?? has a higher value of \( \pi_r \).

![Fig. 1. The manufacturer’s and retailer’s iso-profit curves for a given value of \( a = a^* \).](image-url)
iso-profit curve \( \pi^* \) and below the manufacturer’s iso-profit curve \( \pi_m^* \) such that \( \pi_m > \pi_m^* \) and \( \pi_m > \pi^*_t \) simultaneously.

Referring to Fig. ??, we have the following results: \( \pi_m = \pi_m^* \) and \( \pi_m = \pi^*_t \) at points A and F at which the iso-profit curve \( \pi_m^* \) intersects the iso-profit curve \( \pi^*_t \). Since \( t \) takes negative values in the region C–D–F, we have no interest in this region. \( \pi_m > \pi_m^* \) and \( \pi_m = \pi^*_m \) for all points on the arc A–B–C, except point A, of the iso-profit curve \( \pi_m^* \). \( \pi_m > \pi_m^* \) and \( \pi_m = \pi^*_m \) for all points on the arc A–E–D, except point A, of the iso-profit curve \( \pi^*_m \). \( \pi_t > \pi^*_m \) and \( \pi_m > \pi^*_m \) for all points inside the bounded region A–B–C–D–E.

From the above analysis, we have already seen that if the retailer fixes its strategy at \( q = a^* \) which is its optimal strategy at the two-stage game structure, the manufacturer, as the leader, could adjust its strategy of combination of \( q \) and \( t \) to increase both its and retailer’s profits. From the manufacturer’s point of view, he wants to adjust his strategies to increase his profits as much as possible without reducing the retailer’s profit. This can be implemented by solving the following optimization problem:

\[
\begin{align*}
\text{Max } \pi_m &= \rho_m (1 - a^* - q^* - t^*) - ta^* - q \\
\text{s.t. } &\rho_t (1 - a^* - q^* - (1 - t)a^*) = \pi^*_t, \\
&\leq t \leq 1, \quad q \geq 0.
\end{align*}
\]

Solving problem (??) yields the manufacturer’s optimal solution as follows:

\[
\begin{align*}
\tilde{q} &= [\delta^{\gamma+1} - (\rho_m - \gamma \rho_t)^{-\gamma/(\delta+1)}] \\
&\times (\rho_m + \rho_t)^{(\delta+1)/(\delta+\gamma+1)} \\
\tilde{t} &= 1 - \frac{\rho_t (1 + \gamma) - (\rho_m - \gamma \rho_t)^{\gamma/(\delta+1)} (\rho_m + \rho_t)^{-\gamma/(\delta+1)}}{\gamma (\rho_m - \gamma \rho_t)}.
\end{align*}
\]

However, since we suppose the retailer is a rational person, the manufacturer’s strategy change will force him/her to reconsider its strategy because \( a = a^* \) is not its optimal strategy any more. The retailer’s new strategy can be obtained by considering the following problem:

\[
\begin{align*}
\text{Max } \pi_t &= \rho_t (1 - a^* - q^* - (1 - \tilde{t})a) \\
\text{s.t. } &\rho_t (1 - a^* - (1 - \tilde{t})a) = \pi^*_t, \\
&\leq a \leq 1, \quad \tilde{t} \geq 0.
\end{align*}
\]

which results in the retailer’s optimal strategy as

\[
\begin{align*}
t_{\text{max}} &= \frac{\delta^{\gamma+1} - (\rho_m + \rho_t)^{(\delta+1)/(\delta+\gamma+1)} (\rho_m - \gamma \rho_t)^{\gamma/(\delta+1)} (1 + \gamma)^{\gamma/(\delta+1)} - (\rho_m - \gamma \rho_t)^{\gamma/(\delta+1)} (\rho_m + \rho_t)^{\gamma/(\delta+1)}}{((1 + \gamma)(\rho_m + \rho_t)^{(\delta+1)/(\delta+\gamma+1)} - (\rho_m - \gamma \rho_t)^{\gamma/(\delta+1)} (\rho_m + \rho_t)^{\gamma/(\delta+1)})^{1/(\gamma+1)}}.
\end{align*}
\]

At this point, a new equilibrium point \( (\tilde{a}, \tilde{t}, \tilde{q}) \) is obtained between the manufacturer and the retailer. We refer \( (\tilde{a}, \tilde{t}, \tilde{q}) \) as higher order Stackelberg equilibrium. The following theorem tells us that both the manufacturer and the retailer are better off at \( (\tilde{a}, \tilde{t}, \tilde{q}) \) than at \( (a^*, t^*, q^*) \).

**Theorem 2.**

\[
\begin{align*}
q^* \leq \tilde{q}, \quad t^* > \tilde{t}, \quad a^* > \tilde{a}, \quad \pi^*_t > \tilde{\pi}_t, \\
\pi_m^* < \tilde{\pi}_m, \quad \pi^*_M < \tilde{\pi}.
\end{align*}
\]

where “*” and “-” stand for Stackelberg and higher Stackelberg equilibriums.

Referring to Fig. ??, in order to increase \( \pi_m \) without decreasing the retailer’s profit \( \pi_t = \pi^*_t \), the manufacturer should adjust its strategy to \( (\tilde{q}, \tilde{t}) \) such that at this point its iso-profit curve \( \pi_m^* \) is tangent to the retailer’s iso-profit curve \( \pi^*_t \), which is described in Fig. ?? as the point E. It is clear that at the point E, the manufacturer’s profit is higher, the manufacturer’s brand name investment is higher, and the fraction of the manufacturer’s share of the local advertising expenditures is lower than at the point A (Stackelberg equilibrium). The retailer’s reaction on the manufacturer’s adjusted strategy is to reduce the total amount of local advertising expenditures which results in its higher profit than before.

4. Fully coordinated co-op advertising model

In previous section we focused on the equilibrium results for a two-stage game structure. We assumed that the manufacturer as the leader holds extreme power and has almost complete control over the behavior of the retailer. The retailer is presumably powerless to influence the manufacturer. The relationship is that of an employer and an employee. The fact that, in many industries, manufacturing and retailing are vertically separated makes the effective implementation of the manufacturer’s co-op advertising program difficult. The manufacturer who can best implement a national advertising campaign to promote brand awareness may not know the local market and the retailer’s advertising behavior. The manufacturer can only provide such a co-op advertising program to the retailer and the retailer who knows the local market and can best advertise to create immediate sales decides whether the offer is taken advantage of and to what extent. In other words, it is the retailer, not the manufacturer, decides how much, if any, of the manufacturer’s money is spent.

Recent studies in marketing have demonstrated that in many industries retailers have increased their power relative to the manufacturers’ power over the last two decades. Manufacturers that had dominated their retailers are finding that many retailers now hold the upper hand. For example, in Wal-Mart’s early days, a powerful manufacturer such as Procter and Gamble (P&G) would act, as the leader, and dictate to Wal-Mart how much P&G would sell to them, at what prices, and under what terms. P&G used to limit the quantities of high-demand products they would deliver to Wal-Mart, insist that Wal-Mart carry all sizes of a certain product, and demand that Wal-Mart participate in promotional programs. In turn, Wal-Mart would threaten
to give P&G poorer shelf location or drop its merchandize. There was no information sharing, no joint planning, and no systems coordination [?]. This is an example of the leader–follower game structure discussed in the previous section.

However, Wal-Mart has grown to the point where its stores’ revenues are three times those of P&G company. This is called the Wal-Mart effect named after the retailing giant Wal-Mart who started this phenomenon [?—?]. As Wal-Mart grew, its relationships with P&G evolved into partnership and full coordination. The coordination between Wal-Mart and P&G has accelerated the development of sophisticated systems such as Just-In-Time delivery, Electronic Data Interchange, and Efficient-Consumer-Response systems. These systems have been used by P&G to monitor sales in Wal-Mart stores as well as to promote and ship their goods in response to actual consumer demand [?—?]. Coordination has created value for consumers in the form of lower prices and greater availability of their favorite P&G merchandize. Through coordination, Wal-Mart and P&G have accelerated the development of systems that can monitor the entire supply chain from the earliest market changes, better design and manufacturing processes, inventory control systems, and inter-organizational networks [?]. This coordination and the re-purposing of the manufacturer and retailer’s systems have been used to create a cooperative advertising scheme that maximizes the joint profits of both parties.

Since \( \pi_m \) and \( \pi_r \) are quasi-concave, the set of Pareto efficient schemes consists of those points where the manufacturer’s and the retailer’s iso-profit surfaces are tangent to each other, i.e.,

\[
\nabla \pi_m(a, t, q) + \mu \nabla \pi_r(a, t, q) = 0,
\]

for some \( \mu \geq 0 \), where \( \nabla \pi_m = [\partial \pi_m / \partial a, \partial \pi_m / \partial t, \partial \pi_m / \partial q] \) stands for the gradient of \( \pi_m \). This leads to the following theorem.

**Theorem 3.** The collection of Pareto efficient schemes is described by the set

\[
Y = \{(\bar{a}, t, \bar{q}): 0 \leq t \leq 1\},
\]

where \( \bar{a} = [\delta_{r_m} \delta_{r_t} (\rho_m + \rho_t)]^{1/(\gamma + 1)} \) and \( \bar{q} = [\delta_{r_m} \delta_{r_t} (\rho_m + \rho_t)]^{1/(\gamma + 1)} \).

This theorem tells us that all Pareto efficient schemes are associated with a single local advertising expenditure \( \bar{a} \) and a single manufacturer’s brand name investment \( \bar{q} \) with and with the fraction \( t \) of the manufacturer’s share of the local advertising expenditures between 0 and 1. The locus of tangency lies on a vertical line segment at \( (\bar{a}, \bar{q}) \) in \( (a, t, q) \) space because the expressions for both iso-profit surfaces contain only linear fraction variable \( t \), so that vertically shifting an iso-profit surface yields another iso-profit surface. When a pair of tangent iso-profit surfaces shift vertically, the tangent point also shifts vertically so that the locus of tangency traces a vertical line.

**Theorem 4.** An advertising scheme is Pareto efficient if and only if it is an optimal solution of

\[
\tilde{\pi} = \max_{a, t, q} \pi = \pi_m + \pi_r
\]

s.t. \( 0 \leq t \leq 1, q \geq 0, a \geq 0 \).

This theorem tells us that, among all possible advertising schemes, the system profit (i.e., the sum of the manufacturer’s and the retailer’s profits) is maximized for every Pareto efficient scheme, but not for any other schemes. The following theorem implies that Pareto efficiency yields (1) higher system profit than at both Stackelberg and higher Stackelberg equilibriums, (2) higher manufacturer’s brand name investment than at Stackelberg and lower Stackelberg equilibrium and lower manufacturer’s brand name investment than at higher order Stackelberg equilibrium, and (3) higher local advertising expenditures than at both Stackelberg and higher order Stackelberg equilibrium.

**Theorem 5.**

\[
\tilde{\pi} > \hat{\pi}, \quad \tilde{q} > \bar{q} > q^*, \quad \tilde{a} > a^* > \bar{a},
\]

where “\( \ast \)”, “\( \hat{\ast} \)”, “\( \bar{\ast} \)” and “\( \tilde{\ast} \)” represent coordination, higher order Stackelberg and Stackelberg equilibriums, respectively.
Theorem 2 leads to the possibility that both the manufacturer and the retailer can gain more profits compared with Stackelberg equilibriums. It should be noted that not all Pareto efficient schemes are feasible to both the manufacturer and the retailer. Neither the manufacturer nor the retailer would be willing to accept less profits at full coordination than at Stackelberg equilibriums. An advertising scheme \((\vec{a}^*, t, \vec{q}^*) \in Y\) is called feasible Pareto efficient if
\[
\Delta p_m(t) = p_m(\vec{a}^*, t, \vec{q}^*) - p_m^* \geq 0
\]  
and
\[
\Delta p_r(t) = p_r(\vec{a}^*, t, \vec{q}^*) - p_r^* \geq 0,
\]
since only schemes satisfying (??) and (??) are acceptable for both the manufacturer and the retailer when they do coordinate. We then call
\[
Z = \{ (\vec{a}^*, t, \vec{q}^*) : \Delta p_m(t) \geq 0, \quad \Delta p_r(t) \geq 0, \quad (\vec{a}^*, t, \vec{q}^*) \in Y \},
\]
the feasible Pareto efficient set of advertising schemes. Let
\[
k_1 = \rho_m ((\vec{a}^*)^\gamma (\vec{q}^*)^\delta - (\vec{a}^*)^\gamma (\vec{q}^*)^{-\delta} - (q^* - \vec{q}^*)) + a^* t^*,
\]
\[
k_2 = \rho_r ((\vec{a}^*)^\gamma (\vec{q}^*)^\delta - (\vec{a}^*)^\gamma (\vec{q}^*)^{-\delta} - (a^* - \vec{a}^*) + a^* t^*),
\]
\[t_{\text{min}} = -k_2/\vec{a}^*,
\]
and
\[t_{\text{max}} = k_1/\vec{a}^*.
\]
Here we suppose \(k_2 < 0\). Then \(\Delta p_m(t) = k_1 - \vec{a}^* t, \Delta p_r(t) = k_2 + \vec{a}^* t, \text{ and } Z\) can be simplified as
\[
Z = \{ (\vec{a}^*, t, \vec{q}^*) : t_{\text{min}} \leq t \leq t_{\text{max}} \}. \tag{29}
\]
It can be shown that \(1 > t_{\text{max}} > t_{\text{min}} \geq 0\) (see Appendix A). Therefore, for any given \(t\) which satisfies \(t_{\text{min}} < t < t_{\text{max}}, \Delta p_m(t) > 0\) and \(\Delta p_r(t) > 0\). This simply implies that there exist Pareto efficient advertising schemes such that both the manufacturer and the retailer are better off at full coordination than at Stackelberg equilibriums. We are interested in finding an advertising scheme in \(Z\) which will be acceptable to both the manufacturer and the retailer. According to Theorem 2, for any Pareto scheme \((\vec{a}^*, t, \vec{q}^*)\),
\[
\Delta p_m(\vec{a}^*, t, \vec{q}^*) + \Delta p_m(\vec{a}^*, t, \vec{q}^*) = \Delta \pi \text{ where } \Delta \pi = \vec{a}^* - \vec{a}^* \text{ is a positive constant. We refer } \Delta \pi \text{ as the system profit gain since it is the joint profit gain achieved by the manufacturer and the retailer by moving from a Stackelberg advertising scheme to a Pareto efficient advertising scheme. This property implies that the more the manufacturer’s share of the system profit gain, the less the retailer’s share of the system profit gain, and vice versa. The property that all feasible efficient transactions occur at \((\vec{a}^*, \vec{q}^*)\) implies that the manufacturer and the retailer will agree to change the local advertising expenditures to \(\vec{a}^*\) from \(a^*\) and the brand name investments to \(\vec{q}^*\) from \(q^*\). However, they will negotiate over the manufacturer’s share of the local advertising expenditures \(t\).

Assume that the manufacturer and the retailer agree to change local advertising expenditures to \(\vec{a}^*\) and brand name investments to \(\vec{q}^*\) from \(a^*\) and \(q^*\), respectively, and engage in bargaining for the determination of reimbursement percentage to divide the system profit gain. A fraction closer to \(t_{\text{max}}\) is preferred by the retailer, and a fraction closer to \(t_{\text{min}}\) is preferred by the manufacturer. As we know, it is impossible to determine the division of the system profit gain without any further assumptions (see, for example [2,7]).

Suppose both the manufacturer and the retailer have preferences for the amount of shares of the system profit gain, which preferences are represented by each system member’s von Neumann–Morgenstern [7] (vN–M) cardinal utility function for \((\Delta p_m, \Delta p_r)\). The manufacturer’s and the retailer’s utility functions are denoted by \(u_m(\Delta p_m, \Delta p_r)\) and \(u_r(\Delta p_m, \Delta p_r)\), respectively. These utility functions can be assessed by considering preferences among lotteries involving \((\Delta p_m, \Delta p_r)\). The system utility function is based on both system members’ utility functions and their bargaining positions. The detail of assessment of the members’ individual and system’s (vN-M) utility functions could be found in, for example [2,7].

There are different forms of individuals’ utility functions. The most appropriate type of multiattribute utility function in realistic applications of decision analysis is an additive form. A utility function is said to be additive if it can be written in the form
\[
u_i(\Delta p_m, \Delta p_r) = \sum_{j=m,r} b_{ij} u_j(\Delta \pi_j) \quad \text{for } i = m, r, \tag{30}
\]
where \(u_j\) is the conditional utility function of member \(i (i = m, r)\) for \(\Delta \pi_j (j = m, r)\) (which is assumed to be a monotonic and increasing function of \(\Delta p_i\)), and \(b_{ij}\) is a positive scaling constant with \(\sum_{j=m,r} b_{ij} = 1 \text{ for } i = m, r\). A utility function is additive if and only if \(\Delta p_m\) and \(\Delta p_r\) are additive independent, which means that preferences over lotteries involving \(\Delta p_m\) and \(\Delta p_r\) depend only on their marginal probability distributions [7]. It has been also shown that, for additive individual utility functions, the system utility function, \(u_s(\Delta p_m, \Delta p_r)\), is also additive under the linear aggregation rule. The form of \(u_s(\Delta p_m, \Delta p_r)\) is as follows:
\[
u_s(\Delta p_m, \Delta p_r) = \lambda_m u_m(\Delta p_m, \Delta p_r) + \lambda_r u_r(\Delta p_m, \Delta p_r), \tag{31}
\]
where \(\lambda = (\lambda_m, \lambda_r) \geq 0\) is the vector of aggregation weights which reflect the relative power or importance of the manufacturer or the retailer in the bargaining table and \(\lambda_m + \lambda_r = 1\). In this general additive form, each member is assumed to have preferences not only for its own shares but also for the partner’s shares. A degenerate additive form where each member cares only for its own shares,
and retailer. There are several implications of (??) and (??). First, the proportional divisions of the system profit gain to the manufacturer and the retailer depend only on the two risk aversion measures \( b_m \) and \( b_t \), not on the aggregation weights \( \lambda_m \) and \( \lambda_t \). A member who has higher degree of risk aversion measurement will receive less shares of the system profit gain. For example, suppose \( b_m > b_t \), then the manufacturer’s share of the system profit gain, \( s = b_t/(b_m + b_t) \), is less than the retailer’s share of the system profit gain, \( 1 - s = b_m/(b_m + b_t) \). Second, the compensation fee, \( \omega \), depends on both the aggregation weights as well as the risk aversion measurements. If \( \lambda_m/\lambda_t > b_t/b_m \), an increase in \( \lambda_m \) or a decrease in \( \lambda_t \) will cause an increase of compensation fee, \( \omega \), from the retailer to the manufacturer, and a decrease of \( \lambda_m \) or an increase of \( \lambda_t \) will decrease the compensation fee, \( \omega \), from the retailer to the manufacturer. In other words, the more power the manufacturer has, the higher compensation fee he/she receives from the retailer. Similar analysis can be done for the case where \( \lambda_m/\lambda_t < b_t/b_m \). Third, if both the manufacturer and the retailer have the same degree of risk aversion measurements, i.e., \( b_m = b_t = b \), then \( s = 1/2 \) and \( \omega = (1/2b) \log(\lambda_m/\lambda_t) \). This means that they equally divide the system profit gain and whether a member receives a positive or negative compensation fee from the other member depends only on its bargaining power. A member who has more bargaining power will receive a positive compensation, and vice versa.

Since on the Pareto frontier \( \Delta \pi_m(t) = k_1 - \bar{a}^*t \) and \( \Delta \pi_t(t) = k_2 + \bar{a}^*t \) with \( t_{\text{min}} \leq t \leq t_{\text{max}} \), the best Pareto advertising reimbursement, \( t \), is in the intersection between the Pareto frontier and the utility related divisions (??) and (??), i.e.,

\[
\Delta \pi_m(t) = \Delta \pi_m^* \quad \text{and} \quad \Delta \pi_t(t) = \Delta \pi_t^*.
\]

Solving \( t \) yields

\[
\bar{t} = \max - \Delta \pi_m^*/\bar{a}^* = \min + \Delta \pi_t^*/\bar{a}^*,
\]

which satisfies \( t_{\text{min}} \leq \bar{t} \leq t_{\text{max}} \).

This example illustrates several important properties of transactions negotiated by the manufacturer and the retailer. First, if the manufacturer and the retailer have the same degree of risk aversion measures, the model suggests that the members should equally share the system profit gain. Second, the model yields the result that the more risk averse a member, the lower the share of the system profit gain received by the member. Finally, the model predicts that the member who has higher bargaining power will receive a positive compensation fee from the other member.

5. Concluding remarks

This paper investigates cooperative (co-op) advertising roles in a manufacturer–retailer supply chain. First, we discuss the classical co-op advertising models in the literature. In this classical model, the solution of game is called “Stackelberg” equilibrium. Next, we develop a new-game structure
called as “higher order Stackelberg” equilibrium in the text of co-op advertising. Comparisons of these two-game structures are discussed. Finally, we develop a fully coordinated game model to explore current marketing phenomenon, that is, retailers have equal or more power than a manufacturer. A cooperative bargaining model is utilized to select the best co-op advertising expenditure sharing rule between the manufacturer and the retailer taking their preferences and risk attitudes into accounts.

There are two possible avenues for future research. First, the single manufacture–retailer system assumption can be relaxed to a duopoly structure of manufacturers who sell their products through a common monopolistic retailer who sells multiple competing brands with varying degrees of substitutability. It would be interesting to discuss the impact of duopolistic manufacturers’ brand name investments, monopolistic retailer’s local advertising level, and advertising sharing rules among manufacturers and retailer on co-op advertising expenditures. Second, in our analysis we employed nonlinear sales response function to satisfy the saturation requirement. As indicated in the literature of channel studies, many important results in equilibrium analyses depend on the shape of the product demand function [?]. Therefore, the use of a linear sales response function may yield different and interesting results in the analysis for vertical co-op advertising agreements.

\[
\pi_t = \rho_t \left(1 - \bar{a} \bar{q}^{-\delta} \right) - (1 - \bar{q}) \bar{a} \\
= \rho_t (1 + \gamma) \rho_t \left[ (1 + \gamma) (\rho_m + \rho_t) ^{(\delta + 1)} \left( \left( \rho_m - \gamma \rho_t \right) \left( \rho_m - \gamma \rho_t \right) \right) ^{(\delta + 1)} \left( \left( \rho_m - \gamma \rho_t \right) \left( \rho_m - \gamma \rho_t \right) \right) ^{(\delta + 1)} \right] ^{1/(\gamma + 1)}
\]

\[
(A.7)
\]

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Appendix A. Proof of results

Proof of Theorem ???. Solving the first-order conditions of \( \pi_m \) with respect to \( t \) and \( q \) yields

\[
t^* = \frac{\rho_m - (1 + \gamma) \rho_t}{\rho_m - \gamma \rho_t}
\]

\[
(A.1)
\]

and

\[
q^* = \left[ (\gamma + 1) ^{(\delta + 1)} \right] ^{1/(\gamma + 1)}
\]

\[
(A.2)
\]

Substitute (??) and (??) into (??), we have

\[
a^* = \left[ (\gamma + 1) ^{(\delta + 1)} \right] ^{1/(\gamma + 1)}
\]

\[
(A.3)
\]

Hence, \((a^*,t^*,q^*)\) is the optimal solution of (??).

Proof of Theorem ?.?. Substituting (??)–(??) into (??) and (??), we have

\[
\pi_m^* = \rho_m - [(\gamma + \delta + 1) \rho_m - \gamma (\gamma + \delta + 1) \rho_t] \times \left[ \delta ^{\delta + 1} \left( \rho_m - \gamma \rho_t \right) \right] ^{(\delta + 1)} \times 1/(\gamma + 1)
\]

\[
(A.4)
\]

\[
\pi_t^* = \rho_t (1 + \gamma) \rho_t \left[ \delta ^{\delta + 1} \left( \rho_m - \gamma \rho_t \right) \right] ^{(\delta + 1)} \times 1/(\gamma + 1)
\]

\[
(A.5)
\]

Substituting (??), (??), and (16) into (??) and (??), we have

\[
\pi_m = \rho_m \left[ 1 - \bar{a} ^{-\gamma} \bar{q} ^{-\delta} \right] - \bar{q} \bar{a}
\]

\[
= \rho_m - (\rho_m + \rho_t) ^{-\delta(\delta + 1)}
\]

\[
\times \left[(1 + \gamma) (\rho_m + \rho_t) ^{(\delta + 1)} - (\rho_m - \gamma \rho_t) ^{(\delta + 1)} \right] ^{1/(\gamma + 1)}
\]

\[
\times \left[ \delta ^{\delta + 1} \left( \rho_m - \gamma \rho_t \right) \left( \rho_m - \gamma \rho_t \right) ^{(\delta + 1)} \right] ^{(\delta + 1)}
\]

\[
\times \left[ 1 + \gamma (\rho_m + \rho_t) ^{(\delta + 1)} \left( \rho_m + \rho_t \right) ^{(\delta + 1)} - (\rho_m - \gamma \rho_t) ^{(\delta + 1)} \right] ^{1/(\gamma + 1)}
\]

\[
\times \left[ (\gamma + 1) (\rho_m + \rho_t) ^{(\delta + 1)} - (\rho_m - \gamma \rho_t) ^{(\delta + 1)} \right] ^{1/(\gamma + 1)}
\]

\[
(A.6)
\]

\[
(f = (\gamma + \delta + 1) \left[ \delta ^{\delta + 1} \left( \rho_m - \gamma \rho_t \right) \right] ^{(\delta + 1)}
\]

\[
(A.8)
\]

\[
g = \left[ \delta ^{\delta + 1} \left( \rho_m - \gamma \rho_t \right) \right] ^{(\delta + 1)} \times 1/(\gamma + 1)
\]

\[
(A.9)
\]

\[
\Phi(x) = x ^{(\delta + 1)} \left[ 1 + \gamma (\rho_m + \rho_t) ^{(\delta + 1)} - (\rho_m - \gamma \rho_t) ^{(\delta + 1)} \right] ^{1/(\gamma + 1)}
\]

\[
(A.10)
\]

Then \( \pi_m - \pi_m^* = f - g \Phi(\rho_m + \rho_t) \). It is easy to check that \( f - g \Phi(\rho_m - \gamma \rho_t) = 0 \). Since \( \Phi'(\rho_m - \gamma \rho_t) = 0 \) and \( \Phi(x) < 0 \) when \( x > (\rho_m - \gamma \rho_t) \), we have

\[
\Phi(\rho_m + \rho_t) < \Phi(\rho_m - \gamma \rho_t).
\]

Hence

\[
f - g \Phi(\rho_m + \rho_t) > f - g \Phi(\rho_m - \gamma \rho_t) = 0,
\]

\[
i.e. \pi_m > \pi_m^*.
\]

(A.12)
Proof of Theorem ??.

\( \Phi(x) = \left[ y^{-(\gamma + \delta + 1)}(\rho_m - \gamma \rho_t)x^{\delta(\delta+1)}(\gamma + 1)\right]^{1/(\gamma + \delta + 1)} \)

\[ \frac{\Phi(\rho_m + \rho_t)}{x^{\delta(\delta+1)}} = \frac{(\gamma + 1)x^{\delta(\delta+1)} - (\rho_m - \gamma \rho_t)x^{\delta(\delta+1)}}{x^{\delta(\delta+1)}}. \]  

(A.13)

Then

\[ \bar{n}_t - n_t^* = (\gamma + 1)\rho_t(\delta - (\rho_m - \gamma \rho_t)x^{-(\gamma + \delta)} - 1)^{1/(\gamma + \delta + 1)} \times \frac{\Phi(\rho_m + \rho_t)}{x^{\delta(\delta+1)}} \times \Phi(\rho_m + \rho_t). \]  

(A.14)

Since \( \Phi'() = 0 \) and \( \Phi'(x) > 0 \) for any \( x > \rho_m - \gamma \rho_t \), \( \Phi(x) \) is an increasing function for \( x \geq \rho_m - \gamma \rho_t \). Since \( \Phi(\rho_m - \gamma \rho_t) = 0 \) and \( \Phi(x) > 0 \) for any \( x > \rho_m - \gamma \rho_t \), we have

\( \Phi(\rho_m + \rho_t) > \Phi(\rho_m - \gamma \rho_t) = 0 \) i.e., \( \bar{n}_t > n_t^* \).  

(A.15)

Combining (??) and (??), we have \( n_t > n_t^* \).

(iii) Proof of \( \bar{q} > q^* \):

\[ \Phi(x) = (\rho_m + \rho_t) - (\rho_m + \rho_t)^{-(\delta + 1)}(\gamma + 1)(\rho_m + \rho_t)^{\delta(\delta+1)} - x^{\delta(\delta+1)} \]

\[ \times [\delta + x^{-(\gamma + \delta + 1)}(\gamma + 1)]^{1/(\gamma + \delta + 1)} \times \left\{ \frac{1 + x^{-(\gamma + \delta + 1)}(\gamma + 1)}{(\rho_m + \rho_t)^{\delta(\delta+1)}} \right\} \]

\[ \times \left\{ 1 - \frac{\rho_m - \gamma \rho_t}{(\rho_m + \rho_t)} \right\}^{1/(\gamma + \delta + 1)} \]  

(A.21)

\[ \bar{q} - q^* = [\delta^{-\gamma} - \gamma (\rho_m - \gamma \rho_t)](\rho_m + \rho_t)^{\gamma(\gamma + \delta + 1)} \times \left\{ \frac{1 - x^{-\gamma(\gamma + \delta + 1)}}{(\rho_m + \rho_t)^{\gamma(\gamma + \delta + 1)}} \right\} \times \left\{ 1 - \frac{(\rho_m - \gamma \rho_t)}{(\rho_m + \rho_t)} \right\}^{1/(\gamma + \delta + 1)} \]

\[ > 0 \]  

i.e., \( \bar{q} > q^* \).  

(A.16)

(iv) Proof of \( t^* > \bar{t} \):

\[ t^* - \bar{t} = \rho_t \left( 1 - \frac{(\rho_m - \gamma \rho_t)(\rho_m + \rho_t)^{\delta(\delta+1)}}{\rho_m - \gamma \rho_t} \right) > 0 \]  

i.e., \( t^* > \bar{t} \).  

(A.17)

(v) Proof of \( a^* > \bar{a} \):

\[ a^* - \bar{a} = [\delta^{-\gamma} - \gamma (\rho_m - \rho_t)]^{1/(\gamma + \delta + 1)} \times \left\{ 1 - \frac{\rho_m - \gamma \rho_t}{(\rho_m + \rho_t)} \right\}^{1/(\gamma + \delta + 1)} \]

\[ > 0 \]  

i.e., \( a^* > \bar{a} \).  

(A.18)

Proof of Theorem ??.

Since

\[ \nabla_n(a, t, q) = (\gamma \rho_m a^{-\gamma(\gamma + 1)} - t, -a, -a, \rho_m a^{-\gamma(\gamma + 1)} - 1) \]  

(A.19)

and

\[ \nabla_n(a, t, q) = (\gamma \rho_m a^{-\gamma(\gamma + 1)} - t, -a, -a, \rho_m a^{-\gamma(\gamma + 1)} - 1) \]

(A.20)

utilizing (??) we can get \( \mu = 1, a^* \) and \( q^* \) in (??) and with \( t \) between 0 and 1.

Proof of Theorem ??.

Since \( \pi = (\rho_m + \rho_t)(a - a^*-\gamma q^* - a - q) \) does not contain the variable \( t \), any value of \( t \) between 0 and 1 can be a component for any optimal solution.

Taking the first derivatives of \( \pi \) with respect to \( a \) and \( q \), and setting them to 0, we have \( \bar{a}^* = \delta^\gamma, \bar{q}^* = [\delta^\gamma a^{-\gamma}(\rho_m + \rho_t)]^{1/(\gamma + \delta + 1)} \).

Therefore, \( a^*, t, q^* \) for any \( t \) in \([0, 1]\) is an optimal solution of (??).

Proof of Theorem ??.

The relationship between (??) and (*) has been established in Theorem ??.

(i) Proof of \( \bar{n}_t > \bar{n} \):

Let
\[ q^* - q^* = \left[ \delta^{t+1} \gamma^{-1} (\rho_m + \rho_r) \right]^{1/(\gamma + \delta + 1)} \times \left[ 1 - \left( (\rho_m - \gamma \rho_r)/(\rho_m + \rho_r) \right)^{1/(\gamma + \delta + 1)} \right] \]
\[ > 0. \] \( \text{(A.25)} \)

(iii) Proof of \( \tilde{a}^* \geq \tilde{a} \):
\[ \tilde{a}^* - \tilde{a} = \left[ \delta^{t^{\prime}+1} \gamma^{t+1} (\rho_m + \rho_r) \right]^{1/(\gamma + \delta + 1)} \times \left[ 1 - \left( (\rho_m - \gamma \rho_r)/(\rho_m + \rho_r) \right)^{1/(\gamma + \delta + 1)} \right] \]
\[ > 0. \] \( \text{(A.26)} \)

Proof of 1 > \( t_{\text{max}} > t_{\text{min}} \geq 0 \). Since \( t_{\text{max}} - t_{\text{min}} = (k_1 + k_2)/\tilde{a}^* = \Delta t/\tilde{a}^* > 0 \), we have \( t_{\text{max}} > t_{\text{min}} \). Now let us show \( t_{\text{max}} < 1 \). Let
\[ \phi(x) = (\gamma + \delta) x^{1/(\gamma + \delta + 1)} + \frac{\rho_m}{\rho_m - \gamma \rho_r} x^{1/(\gamma + \delta + 1)} \]
\[ - (\gamma + \delta + 1). \] \( \text{(A.27)} \)

Since
\[ \phi'(x) = \frac{\gamma + \delta}{\gamma + \delta + 1} x^{-2(\gamma + \delta + 1)/(\gamma + \delta + 1)} \left( x - \frac{\rho_m}{\rho_m - \gamma \rho_r} \right) > 0 \]
\[ \text{when } x > \frac{\rho_m}{\rho_m - \gamma \rho_r}, \] \( \text{(A.28)} \)

we have
\[ \phi \left( \frac{\rho_m + \rho_r}{\rho_m - \gamma \rho_r} \right) > \phi \left( \frac{\rho_m}{\rho_m - \gamma \rho_r} \right). \] \( \text{(A.29)} \)

After rearrangement of above inequality, we have \( t_{\text{max}} < 1 \). \( \square \)

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